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REPORT NO. 88

THE DECISION AND SYNTHESIS PROBLEMS
IN SEMI-MODULAR SWITCHING THEORY

by

James H. Shelly

May 20, 1959

(This is being submitted in partial
fulfillment of the requirements for
the degree of Doctor of Philosophy
in Mathematics, June 1959.)

This work was supported in part by the
Office of Naval Research under Contract Nonr-1834(27).



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CHAPTER I

INTRODUCTION

Switching theory is the study of systems whose behavior may be described by finite sets of variables whose ranges are discrete sets. This subject was first placed on a firm mathematical foundation by C. E. Shannon, [1]^{*}, in 1938 with the application of Boolean algebra to the analysis of relay devices. Since that time, many efforts have been made to extend and improve mathematical techniques for the analysis and synthesis of switching circuits which utilize many different types of components or elements. The search for efficient and effective analysis and synthesis procedures in switching theory becomes important as the size and complexity of switching circuits increase. Among the tools employed are graph theory [2, 3], Boolean matrix theory [4, 5], and combinatorial topology [6] .

Switching circuits are usually classified as being synchronous or asynchronous, but the definitions of synchronous and asynchronous vary from author to author. The unifying characteristic of most definitions of a synchronous circuit is that the essential behavior of the circuit may be abstracted by examining suitable variables at predetermined instants of time. Any switching circuit which fails this test is classified as asynchronous.

I. THE PROBLEM

Statement of the problem. The aim of this paper is to investigate the decision and synthesis procedures for semi-modular circuits, a sub-class of asynchronous switching circuits. This includes a characterization of such

* Numbers in brackets refer to footnotes at the end of the thesis.

circuits by means of properties of a set of numerical vectors and an application of these results to the synthesis problem.

Importance of the study. An effective synthesis procedure is necessary for the evaluation of semi-modular circuitry in solving actual problems in circuit design. Among the advantages which semi-modular circuitry may possess are the simplification of timing problems caused by finite size of the network and finite signal transit times, elimination of worst-case time considerations, and possible self-checking features at no increase in cost.

Scope and limitations of the study. The synthesis procedure studied in this paper assumes that the description of the circuit to be synthesized is in the form of a set L of n -dimensional vectors of non-negative integers. The object of the synthesis is to produce a semi-modular circuit C whose C -state lattice coincides with L . This is achieved in a direct form in Chapter IV.

Another aspect of the synthesis problem is that the description of the circuit is often in a flow chart form, rather than as a lattice of vectors. The relaxation of this restriction on the circuit description forms another important part of Chapter V.

Within the semi-modular circuit theory, natural limitations on the circuits are present. The determinacy necessary to satisfy the semi-modularity condition means that input sequences and external conditions cannot affect the operation of the circuit. The extensions of the theory to approximate such conditions lies outside the bounds of this paper.

II. NOTATION

This paper is based on an extensive examination of properties of sets of numerical vectors, and it is important to develop a precise and concise notation for displaying the concepts which are employed.

Definitions 1.1 through 1.4 and 1.6 are common in the mathematical literature. Definition 1.5 introduces the notation used for sets of integers which define the spaces discussed in this paper. Definitions 1.7 through 1.10 define operations and relations on these spaces, while the theorems state the properties of these operations.

The proofs of the theorems of this section are immediate, and therefore are omitted.

Set Theory. The basic set-theoretic notation is reviewed in Definition 1.1.

Definition 1.1. If A is any set, $a \in A$ is defined to mean that a is an element of the set. If A and B are subsets of a set $\mathcal{U}$, then $A \subseteq B$ is defined to mean that $a \in A$ implies that $a \in B$. The relation $A = B$ is defined to mean that $a \in A$ if and only if $a \in B$. The relation $A \subset B$ is defined to mean that $a \in A$ implies that $a \in B$, but there exists at least one element b in B such that b is not an element of A . The set $A - B$ consists of all elements of A which are not in B . The void set is denoted by $\emptyset$. If R is any relation between two sets A and B , or between an element a and a set B , then $A \nsubseteq B$ or $a \nsubseteq B$ is the denial or negation of the relation R .

Definition 1.2. If θ is a relation between elements of a set $\mathcal{U}$, then θ is defined to be a partial ordering, if:

1. For each a in $\mathcal{U}$, $a \theta a$, (Reflexive)
2. For any elements a and b in $\mathcal{U}$ such that $a \theta b$ and $b \theta a$,
then $a = b$, (Anti-symmetric)
3. For any elements, a , b , and c in $\mathcal{U}$ such that $a \theta b$ and $b \theta c$,
then $a \theta c$. (Transitive)

The concept of partial ordering leads to a concept of covering, in the following sense.

Definition 1.3. Let $\mathcal{L}$ be a space $\mathcal{L}$ partially ordered under θ . Then a covers b is defined to mean that $a \theta b$, and for any c in $\mathcal{L}$ for which $a \theta c \theta b$, either $a = c$ or $c = b$.

Another consequence of the definition of partially ordered sets is the concept of a lattice.

Definition 1.4. Let $\mathcal{L}$ be a set partially ordered under θ . Then $\mathcal{L}$ is defined to be a lattice if, whenever a and b are elements of $\mathcal{L}$:

1. There exists an element $a \cup b$ of $\mathcal{L}$ such that $a \theta a \cup b$, $b \theta a \cup b$, and if c is in $\mathcal{L}$, $a \theta c$ and $b \theta c$, then $a \cup b \theta c$.
2. There exists an element $a \cap b$ of $\mathcal{L}$ such that $a \cap b \theta a$, $a \cap b \theta b$, and if d is in $\mathcal{L}$, $d \theta a$, and $d \theta b$, then $d \theta a \cap b$.

If $\mathcal{L}$ is a lattice, it is semi-modular if all bounded chains are finite and if for any a , b , and c in $\mathcal{L}$, a covers c , b covers c , and $a \neq b$, then $a \cup b$ covers a and b .

The lattice $\mathcal{L}$ is modular if it is semi-modular and for any a^* , b^* and c^* such that c^* covers a^* , c^* covers b^* and $a^* \neq b^*$, then a^* covers $a^* \cap b^*$ and b^* covers $a^* \cap b^*$.

The lattice $\mathcal{L}$ is distributive if $a \cup (b \cap c) = (a \cup b) \cap (a \cup c)$.

Integers and Vectors. In this section, the vector space notation employed in the remainder of the paper is developed. From this point onward, the symbol " n " refers to a positive integer, fixed throughout the particular theorem or definition.

Definition 1.5. For any integers a and b which satisfy $a \leq b$, $I(a, b)$ is defined to be the set of all integers c such that $a \leq c \leq b$.

$I(a, \infty)$ is defined to be the set of all integers c such that $a \leq c$,

and $I(-\infty, b)$ the set of all integers c such that $c \leq b$. The set of all integers is denoted by $I(-\infty, \infty)$. If $K \subseteq I(-\infty, \infty)$ and $a \in I(-\infty, \infty)$, the set $K + a$ is defined by $K + a = \{h \mid h = k + a \text{ for some } k \text{ in } K\}$.

In Definition 1.5, the symbols ∞ and $-\infty$ are not integers.

Definition 1.6. The Cartesian product $\prod_{i \in I(1,n)} A(i)$ of the sets $A(1) \dots A(n)$ is defined to be the set of all n -tuples $(a_1, a_2, \dots, a_n)$, where for each i in $I(1,n)$ $a_i \in A(i)$. If $A(i) = A$ for each i in $I(1,n)$, then $[A]^n$ is defined to be $\prod_{i \in I(1,n)} A(i)$. Thus $[I(0,a)]^n$ is the set of all ordered n -tuples of members of

$I(0,a)$, and the vector $1001 \in [I(0,\infty)]^4$ but $1001 \notin [I(1,\infty)]^4$.

Next, some ordering properties are defined for the space $[I(-\infty, \infty)]^n$.

The basic properties are extensions of the natural ordering in $I(-\infty, \infty)$.

Definition 1.7. Let θ be a relation defined on pairs of elements of $I(-\infty, \infty)$. Then, unless otherwise stated, whenever $\underline{a}$ and $\underline{b}$ are elements of $[I(-\infty, \infty)]^n$, $\underline{a} \theta \underline{b}$ is defined to mean $\underline{a}_i \theta \underline{b}_i$ for each i in $I(1,n)$.

For example, if $\underline{a}$ and $\underline{b}$ are in $[I(-\infty, \infty)]^n$, $\alpha \in I(-\infty, \infty)$ and $K \subseteq [I(-\infty, \infty)]^n$, $\underline{a} + \underline{b}$, $\alpha \underline{a}$, $\underline{a} = \underline{b}$, $\underline{a} \leq \underline{b}$, $K + \underline{a}$ illustrate the componentwise operations described in Definition 1.7. The expression $\underline{a} < \underline{b}$ differs from this form since it is defined by $\underline{a} < \underline{b}$ but $\underline{a} \neq \underline{b}$.

Theorem 1.1. Let θ be a partial ordering of $I(-\infty, \infty)$, and let θ be defined on $[I(-\infty, \infty)]^n$ as in Definition 1.7. Then θ is a partial ordering on $[I(-\infty, \infty)]^n$.

Corollary 1.1. The relations $=$ and $\leq$ of Definition 1.7 are partial orderings in $[I(-\infty, \infty)]^n$.

Thus any subset of $[I(-\infty, \infty)]^n$ is partially ordered by the relation $\leq$.

Several special definitions are applicable in $I(-\infty, \infty)^n$. These include componentwise maximum and minimum operations, and a set of unit vectors.

Definition 1.8. The operations $\vee$ and $\wedge$ are defined on $I(-\infty, \infty)$ by $a \vee b = \max(a, b)$ and $a \wedge b = \min(a, b)$. The element $\underline{0}$ of $[I(-\infty, \infty)]^n$ is defined by $\underline{0}_i = 0$ for each i in $I(1, n)$. The vector $\underline{e}$ of $[I(-\infty, \infty)]^n$ is defined by $\underline{e}_i = 1$ for each i in $I(1, n)$. If $j \in I(1, n)$, the vector $\underline{e}(j)$ of $[I(-\infty, \infty)]^n$ is defined by $\underline{e}_j(j) = 1$, $\underline{e}_i(j) = 0$ for all i in $I(1, n)$ not equal to j .

In formulas such as $\prod_i, \sum_i, \bigcup_i$, the index set for i is understood to be that most recently associated with i . The definitions of $\vee$ and $\wedge$ are understood to extend to $[I(-\infty, \infty)]^n$ by Definition 1.7.

Theorem 1.2. Let $\underline{a}$ and $\underline{b}$ be elements of $[I(-\infty, \infty)]^n$, and α, β be elements of $I(-\infty, \infty)$. The operations $\underline{a} + \underline{b}$ and $\alpha \underline{a}$ are associative and commutative. The distributive laws $\alpha(\underline{a} + \underline{b}) = \alpha \underline{a} + \alpha \underline{b}$ and $(\alpha + \beta) \underline{a} = \alpha \underline{a} + \beta \underline{a}$ hold.

Lemma 1.1. The set $[I(-\infty, \infty)]^n$ is a distributive lattice under $\leq$. Thus the following relations hold in any subset L of $[I(-\infty, \infty)]^n$ which is closed with respect to $\vee$ and $\wedge$.

- | | |
|--|--|
| i) $\underline{a} \vee \underline{a} = \underline{a}$ | v) $\underline{a} \wedge \underline{a} = \underline{a}$ |
| ii) $\underline{a} \vee \underline{b} = \underline{b} \vee \underline{a}$ | vi) $\underline{a} \wedge \underline{b} = \underline{b} \wedge \underline{a}$ |
| iii) $(\underline{a} \vee \underline{b}) \vee \underline{c} = \underline{a} \vee (\underline{b} \vee \underline{c})$ | vii) $(\underline{a} \wedge \underline{b}) \wedge \underline{c} = \underline{a} \wedge (\underline{b} \wedge \underline{c})$ |
| iv) $(\underline{a} \vee \underline{b}) \wedge \underline{a} = \underline{a}$ | viii) $(\underline{a} \wedge \underline{b}) \vee \underline{a} = \underline{a}$ |

The following relations are equivalent.

- ix) $\underline{a} \leq \underline{b}$
- x) $\underline{a} \vee \underline{b} = \underline{b}$
- xi) $\underline{a} \wedge \underline{b} = \underline{a}$

If $(\underline{a} \wedge \underline{b}) \vee (\underline{a} \wedge \underline{c})$ and $\underline{a} \vee (\underline{b} \wedge \underline{c})$ are in L , they are equal.

Lemma 1.2. The distributive laws $(\underline{a} + \underline{b}) \vee (\underline{a} + \underline{c}) = \underline{a} + \underline{b} \vee \underline{c}$ and $(\underline{a} + \underline{b}) \wedge (\underline{a} + \underline{c}) = \underline{a} + \underline{b} \wedge \underline{c}$ hold in $[I(-\infty, \infty)]^n$. If $\underline{a}$ and $\underline{b}$ are elements of $[I(-\infty, \infty)]$, then $\underline{a} + \underline{b} = \underline{a} \vee \underline{b} + \underline{a} \wedge \underline{b}$.

In this paper, parentheses will be omitted in expressions such as

$$\underline{a} + (\underline{b} \vee \underline{c}) \text{ and } (\underline{a} \wedge \underline{b}) + \underline{c}.$$

Lemma 1.3. Let r be an element of $I(1, \infty)$. If for each j in $I(1, r)$, $\underline{b}(j) \in [I(-\infty, \infty)]^n$ and if for each pair of distinct indices i, j in $I(1, r)$, $\underline{b}(i) \wedge \underline{b}(j) = 0$, then
$$\sum_{j \in I(1, r)} \underline{b}(j) = \bigvee_{j \in I(1, r)} \underline{b}(j).$$
 Finally, an extension of the concept of residue classes modulo r ,

where $r \in I(1, \infty)$, is useful in later chapters.

Definition 1.9. Let K be a subset of $[I(-\infty, \infty)]^n$, and let $\underline{a}$ and $\underline{b}$ be elements of $[I(-\infty, \infty)]^n$. Then $\underline{a} = \underline{b} \text{ mod } K$ is defined to mean that there exists a set $\{m_k\}$ of elements of $I(-\infty, \infty)$ such that

$$\underline{a} = \underline{b} + \sum_{k \in K} m_k k.$$

Theorem 1.3. The relation " $= \text{ mod } K$ " is an equivalence relation.

A special class of such equivalence relations is employed in later development. The set K for this class abstracts the non-zero elements of any member of K .

Definition 1.10. If K is any subset of $[I(0, \infty)]^n$, then $M(K)$ is defined by $M(K) = \{j \mid \text{There exists at least one } k \text{ in } K \text{ for which } k_j \neq 0\}$. The set $N(K)$ is defined by $N(K) = \{\underline{e}(j) \mid j \in M(K)\}$. The set $M(K)$ is called the set of component indices spanned by K . The set $\{I(1, n) - M(K)\}$ is the set of unspanned component indices of K .

For example, if $n = 4$ and $K = \{0230, 4020\}$, $M(K) = \{1, 2, 3\}$ and $N(K) = \{1000, 0100, 0010\}$.

III. AN OUTLINE OF SEMI-MODULAR CIRCUIT THEORY

In this section, the basic concepts of semi-modular circuit theory are presented. The material is essentially that of [7] with certain changes in notation to conform with the usage of this paper. The definition of the R-relation in this outline differs from that of the reference, and is obtained from a manuscript to be published later.

The theory of semi-modular circuits is concerned with:

1. n signals z_i , $i \in I(1, n)$, where $z_i \in I(0, k_i - 1)$, and $k_i \in I(1, \infty)$;
2. the set $T = \prod_{i \in I(1, n)} I(0, k_i - 1)$, composed of all ordered n -tuples of the signals;
3. a set of n functions $z'_i = f_i(z_1, \dots, z_n)$, $i \in I(1, n)$, which map T into itself.

The elements of T are called states. In the remainder of the section the set T and $\{f_i\}$ are considered to constitute a circuit C .

The first step in the theory is to characterize those states b which may occur as the images of a given state a , under the transformations defined by the functions f_i . This is accomplished in the R-relation.

Definition 1.11. if x and y are elements of T , then $x R y$ is defined to mean that for each i in $I(1, n)$, $y_i = x_i$ or $y_i = x'_i = f_i(x_1, x_2, \dots, x_n)$.

This definition of the R-relation may be extended to include a sequence of states through which the circuit may pass, by means of a partial allowed sequence.

Definition 1.12. The sequence $\{x(j)\}$, $j \in I(1, r)$ for some r in $I(1, \infty)$, is defined to be a partial allowed sequence if:

1. $x(i) R x(i+1)$
 2. $x(i) \neq x(i+1)$
- } for each i in $I(1, r-1)$.

If $\{x(j)\}$, $j \in I(1,r)$ is a partial allowed sequence, then the relation $\mathcal{H}$ is defined by $x(1)\mathcal{H} x(r)$.

Using Definition 1.12, an allowed sequence is defined as a possibly infinite sequence of states through which the circuit may pass, under the restriction that from some point on, no component is continuously excited but never reaches a new value.

Definition 1.13. The sequence $\{x(j)\}$, $j \in I(1,r)$ or $j \in I(1,\infty)$ is defined to be an allowed sequence if:

1. $\{x(j)\}$ is non-void,
2. $x(i) R x(i+1)$ and $x(i) \neq x(i+1)$ whenever i and $i+1$ are elements of the index set,
3. Whenever there exists an $x(k)$ in $\{x(j)\}$ such that $x(r(1)) = x(r(2))$ and $x'(r(1)) = x'(r(2))$ whenever $x(k) \mathcal{H} x(r(1))$ and $x(k) \mathcal{H} x(r(2))$, then $x'(r) = x(r)$ for any r such that $x(k) \mathcal{H} x(r)$.

The relation $\mathcal{H}$ of definition 1.12 is reflexive and transitive, but not symmetric. This suggests defining an equivalence relation of the form $a \mathcal{H} b$ and $b \mathcal{H} a$ among the elements of T . This is accomplished in Definition 1.14.

Definition 1.14. If x and y are elements of T , then $x \mathcal{E} y$ is defined to mean $x \mathcal{H} y$ and $y \mathcal{H} x$.

Theorem 1.4. The relation $\mathcal{E}$ is an equivalence relation.

Thus the states of C are partitioned into mutually exclusive, jointly exhaustive equivalence classes.

The $\mathcal{H}$ -ordering between states of C induces a corresponding ordering between the equivalence classes, in the following sense:

Definition 1.15. If X and Y are equivalence classes of states of C , or, briefly, equivalence classes of C , then $X \mathcal{H} Y$ is defined to mean that there exist states x in X , y in Y such that $x \mathcal{H} y$.

Theorems 1.5 and 1.6 yield important properties of the $\mathcal{H}$ relation for equivalence classes.

Theorem 1.5. If X and Y are equivalence classes of C for which $x \in X$ and $y \in Y$, and $X \mathcal{H} Y$, then $x \mathcal{H} y$.

Theorem 1.6. The $\mathcal{H}$ relation is a partial ordering between the equivalence sets of C .

In order to clarify restriction 3 of Definition 1.13, three more concepts are needed.

Definition 1.16. A subset G of T is defined to be a natural set if $x \in G$ and $x R y$ implies that $y \in G$. $G(u)$ is the natural set of all x such that $u \mathcal{H} x$.

Definition 1.17. A subset X of T is defined to be a terminal set, if for all members $x(1) \dots x(k)$ of X , there is no index i in $I(1, n)$ such that $x_i(j) = x_i$ and $x'_i(j) = x'_i \neq x_i$ for all states $x(j)$ of X .

Definition 1.18. An equivalence class X is defined to be a final set if $X = G(x)$ for each x in X . If X is a terminal set of C which is not a final set, then X is pseudo-final.

Theorem 1.7. If X is a final set of C , then X is a terminal set.

These three definitions make it possible to state Theorem 1.8 which shows that each allowed sequence has a terminal set, and that for each equivalence class there is at least one final set.

Theorem 1.8. Each allowed sequence $x(1), x(2), \dots$ of elements of C contains an element $x(m)$ such that all $x(j)$ for which $x(m) R x(j)$ are elements of the same terminal class. For each equivalence class Y of C , there is at least one final set F such that $Y \mathcal{H} F$.

The concept of speed-independence is based upon the concept of final set. In the concept, every allowed sequence beginning with a state u contains an element of a fixed final set M .

Definition 1.19. A circuit C is defined to be speed-independent with respect to an initial state u if there exists a final set M such that each allowed sequence beginning with u contains a state of M .

Thus a speed-independent circuit will have some state of each allowed sequence in a final set M , so that the ultimate behavior is independent of the relative order in which the components change.

The semi-modularity condition implies speed-independence, but the converse is not true.

Definition 1.20. A circuit C is defined to be semi-modular with respect to an initial state u if for all states x and y for which $u \not\prec x$ and $x R y$ the relation $y R x'$ holds.

Theorem 1.9. If C is semi-modular with respect to u , then C is speed-independent with respect to u .

The speed-independence property may be verified by examining those pairs of elements of C for which $s R y$, without explicitly constructing all allowed sequences.

If $x(1), x(2), \dots, x(k)$ is any sequence of states such that $x(i) R x(i+1)$, the length vector i of the sequence is defined recursively by $L[x(1)] = (0, 0, \dots, 0)$,

$L[x(i), x(i+1)] = \sum_j \underline{e}(j)$, where the $\underline{e}(j)$ is the j th unit vector of $[I(0, \infty)]^n$ and the summation is taken over those j in $I(1, n)$ for which $x_j(i+1) = x'_j(i) \neq x_j(i)$.

$$L[x(1), \dots, x(i+1)] = L[x(1) \dots x(i)] + L[x(i), x(i+1)].$$

Many properties of the semi-modular circuit may be more easily expressed in terms of the length of partial allowed sequences.

Definition 1.21. A C-state $\underline{a}$ of a partial allowed sequence $x(1), \dots, x(k), x$ of a circuit semi-modular with respect to u (s.m.[u]) is defined to be $L[u, x(1) \dots x(k), x]$. The set of all such C-states is denoted by $C[u]$.

Theorem 1.10 shows that the state $t(\underline{a}) = x$, the terminal state of such a sequence, is independent of the partial allowed sequence by which $\underline{a}$ is defined.

Theorem 1.10. If C is s.m.[u] and $\underline{a}$ is a C-state, then $t(a)$ is unique and independent of the partial allowed sequence used to define $\underline{a}$.

This theorem justifies naming the collection of all C-states whose terminal states lie in some equivalence class, in Definition 1.22.

Definition 1.22. If C is s.m.[u], then the set $\underline{A}$ of all elements $\underline{b}$ of $C[u]$, for which $x = t(\underline{b})$ is an element of an equivalence class X , is defined to be a cyclic class of C .

The pertinent properties contained in the next two theorems are that if $\underline{a}$ and $\underline{b}$ are C-states with initial state u , $\underline{a} \vee \underline{b}$ is a C-state and $C[u]$ is a semi-modular lattice.

Theorem 1.11. $C[u]$ is closed under $\vee$.

Theorem 1.12. $C[u]$ is a semi-modular lattice under $\leq$ with $\underline{0}$ as its least element .

Theorem 1.12 also implies that $C[u] \subseteq [I(0, \infty)]^n$.

Theorem 1.13 is important in the discussion of the synthesis problem.

Theorem 1.13. The elements $\underline{a}$ and $\underline{b}$ of $C[x]$ satisfy $\underline{b}$ covers $\underline{a}$ in $C[u]$ if and only if $\underline{b}$ covers $\underline{a}$ in $[I(0, \infty)]^n$.

The recurrence properties of the states of the circuit may be studied in $C[u]$.

Definition 1.23. If $\underline{v}$ is in $C[u]$, then the sets $D[\underline{v}]$ and $E[\underline{v}]$ are defined by:

$$\begin{aligned} D[\underline{v}] &= \{ \underline{a} \mid \underline{a} \in C[u], \text{ and } \underline{v} \leq \underline{a} \} \\ E[\underline{v}] &= \{ \underline{a} \mid \underline{a} \in D[\underline{v}], \text{ and } t(\underline{a}) = t(\underline{v}) \}. \end{aligned}$$

The basic result is that there exists a basis for $E[\underline{v}]$.

Theorem 1.14. If $\underline{v}$ is in $C[u]$, there exists a set $W[\underline{v}] = \{w(i)\}$, $i \in I(1,k)$, of elements of $[I(0,\infty)]^n$, such that $w(i) + w(j) = w(i) \vee w(j)$, whenever $w(i)$ and $w(j)$ are elements of $W[\underline{v}]$. Furthermore $E[\underline{v}]$ is the totality of vectors of the form

$$\underline{x} = \underline{v} + \sum_{i \in I(1,k)} a(i)w(i), \text{ where for each } i \text{ in } I(1,k), a(i) \in I(0,\infty).$$

Properties of the $\leq$ relation between C-states yield the result that if $\underline{a} \leq \underline{b}$, then each element $w(i)$ of $W[\underline{a}]$ is of the form $w(i) = \sum_j c(j)w(j)$, where $w(j)$ is an element of $W[\underline{b}]$ and $c(j) \in I(0,\infty)$. The application of Theorem 1.14 shows that whenever the terminal states v, v^* of two C-states $\underline{v}, \underline{v}^*$ coincide, the sets $W[\underline{v}]$ and $W[\underline{v}^*]$ coincide, and may be denoted $W[\underline{v}]$. In fact, if $\underline{v}$ and $\underline{v}^*$ are in the same cyclic class $\underline{A}$, then $W[\underline{v}] = W[\underline{v}^*] = W[\underline{A}]$.

Theorem 1.15. If C is s.m.[u] and $\underline{a}$ and $\underline{b}$ are in the same cyclic class, then $W[\underline{a}] = W[\underline{b}]$.

The relations between elements of $W[\underline{A}]$ and elements of $\underline{A}$ are explored in the next theorem.

Theorem 1.16. Let $\underline{A}$ be a cyclic class of C , which is s.m.[u]. If the components of $\underline{a}$ and $\underline{b}$ spanned by $W[\underline{A}]$ are identical, then $\underline{a}$ and $\underline{b}$ both lie in $\underline{A}$. If $\underline{a}$ and $\underline{b}$ both lie in $\underline{A}$, then the components $\underline{a}_i$ and $\underline{b}_i$ unspanned by $W[\underline{A}]$ satisfy $\underline{a}_i = \underline{b}_i$ and $\underline{a}'_i = \underline{b}'_i$.

This concludes the outline of the theory and semi-modular circuits.

Several other features of the theory are generalized in later chapters, and are not included in this outline.

In the remainder of the paper, the blanket hypothesis "Let L be a subset of $[I(-\infty, \infty)]^n$ " is employed. A further notational convention is that a centered horizontal line has the force of such a statement as "This completes the proof of the theorem."

CHAPTER II

RECURRENCE PROPERTIES FOR GENERAL SETS OF VECTORS

In this chapter, the elements of general subsets L contained in $[I(-\infty, \infty)]^n$ are decomposed into equivalence classes which describe the recurrence properties of L . This description is obtained by forming for each element $\underline{a}$ of L the set of vector differences $\underline{b} - \underline{a}$ for all members $\underline{b}$ of L which satisfy $\underline{b} \geq \underline{a}$. The properties of these classes provide the necessary framework on which to construct the concept of periodic class in Chapter III.

The chapter begins with a preliminary theorem concerning the number of minimal elements of a set L which has a lower bound. The $\mathcal{L}$ -relation is defined, and shown to be an equivalence relation. After a study of some of the basic properties of the relation, the recurrence of patterns of differences between elements of L is examined. This investigation leads to the definition of a basis for subsets of the $\mathcal{L}$ -class which is dependent only upon the $\mathcal{L}$ -class, and not upon the choice of the element which defines the subset. The last section of the chapter is devoted to obtaining a sufficient condition that the basis defined above is composed of linearly independent elements.

I. FINITENESS OF THE SET OF MINIMAL ELEMENTS OF SUBSETS OF $[I(c, \infty)]^n$

The result of this section is that any subset H having a lower bound has a finite number of minimal elements, that is elements $\underline{a}$ of H such that no element $\underline{b}$ of H is less than $\underline{a}$. The theorem is an extension of a theorem of Birkhoff [8], and the proof is due to Muller.

Definition 2.1. Let H be a subset of $[I(c, \infty)]^n$, for some c in $I(-\infty, \infty)$. If $\underline{a}$ is an element of H , then $\underline{a}$ is defined to be a minimal element of H if for any $\underline{b}$ in H , $\underline{b} \not< \underline{a}$. The set $R(H)$ is defined to be the set of minimal elements of H .

Theorem 2.1. If $H \subseteq [I(c, \infty)]^n$ for some c in $I(-\infty, \infty)$, then $R(H)$ is finite.

Proof. If K is a subset of $I(1, n)$, let $H(K)$ denote the projection of H onto $[I(c, \infty)]^{n(K)}$ defined by $\underline{a}_i(K) = \underline{a}_i$ if $\underline{a} \in H$, and $i \in K$. The proof is inductive on $n(K)$, the number of elements of K , for $n(K) \leq n$.

If $n(K) = 1$, then $H(K) \subseteq I(c, \infty)$. Since $I(c, \infty)$ and $I(1, \infty)$ are isomorphic under the transformation $a + c = b + 1$, where $a \in I(c, \infty)$ and $b \in I(1, \infty)$, and since $I(1, \infty)$ is well ordered, so is $I(c, \infty)$. Thus $H(K)$ possesses a least element, which is also minimal. This implies that $R(H(K))$ is finite.

The induction hypothesis is that whenever $n(K) \leq r - 1 < n$, $R(H(K))$ is finite. To show that the theorem holds for $n(K) = r$, for each i in K , let $K(i) = K - \{i\}$.

For any member $\underline{a}^*$ of $R(H(K(i)))$, the set of component values $\underline{a}_j$ such that $\underline{a}$ is a member of $H(K)$ and $\underline{a}_j = \underline{a}_j^*$ for all j in $K(i)$, has a minimal element $\underline{a}_i^*$. From the induction hypothesis, the number of such minimal elements $\underline{a}_i^*$ is finite, so that $f_i(K) = \max_{\substack{\underline{a}^* \in R(H(K(i)))}} \underline{a}_i^*$ is defined as an element of $I(c, \infty)$.

If $\underline{d}(K)$ is an element of $H(K)$, then $\underline{d}(K(i)) \geq \underline{a}(K(i))$ for some $\underline{a}(K(i))$ in $R(H(K(i)))$. If in addition, $\underline{d}_i(K) > f_i(K)$, then $\underline{d}(K) > \underline{a}(K)$, and $\underline{d}(K)$ is not a minimal element of $H(K)$. This implies that any minimal element $\underline{d}(K)$ of $H(K)$ must satisfy $\underline{d}_i(K) \leq f_i(K)$ for each i in K . Since $H(K)$ is bounded from below by $(c, c, \dots, c)$, there are at most $\prod_{i \in K} (f_i(K) - c + 1)$ elements in $R(H(K))$. Since this number is finite, the theorem holds for $n(K) = r$, and hence for all $r \leq n$, which includes $K = I(1, n)$.

An immediate result of the theorem is that whenever a subset H of $[I(c, \infty)]^n$ is infinite, there are two elements $\underline{a}$ and $\underline{b}$ of H which are ordered under $<$.

Corollary 2.1. If H is an infinite subset of $[I(c, \infty)]^n$, then there exist elements $\underline{a}$ and $\underline{b}$ of H such that $\underline{a} < \underline{b}$.

Proof. If H is infinite, $H - R$ is non-void whenever R is finite.

Since this is the case when $R = R(H)$, H contains a non-minimal element $\underline{b}$. If $\underline{b}$ is not a minimal element of H , then there is an element $\underline{a}$ in H , such that $\underline{a} < \underline{b}$.

The hypothesis of Theorem 2.1 that H have a lower bound is necessary, since if $H \subseteq [I(-\infty, \infty)]^2$ and consists of the pairs of integers $(i, c-i)$ for any i in $I(0, \infty)$ for some fixed c in $I(0, \infty)$, then no two members of H are ordered, and hence all elements of H are minimal. Since the set is countably infinite, the conclusion of 2.1 does not hold.

II. $L(\underline{a})$ AND THE $\mathcal{S}$ -RELATION

In this section, a binary relation, $\mathcal{S}$, is defined on elements of an arbitrary set L . This relation is defined in terms of a function $L(\underline{a})$ which lists the set of elements $\underline{a} + \underline{b}$ of L which satisfy $\underline{a} + \underline{b} \geq \underline{a}$ for some fixed $\underline{a}$ in L . Theorems 2.2 and 2.3 display two properties of $L(\underline{a})$.

Definition 2.2. Let $\underline{a}$ be an arbitrary member of L . Then $L(\underline{a})$ is defined to be the set $\{\underline{y} \mid \underline{y} + \underline{a} \in L \text{ and } \underline{y} + \underline{a} \leq \underline{a}\}$.

Theorem 2.2. Let $\underline{a}$ and $\underline{y} + \underline{a}$ be members of L . Then $\underline{y} + \underline{a} \geq \underline{a}$ if and only if $\underline{y} \in [I(0, \infty)]^n$.

Proof. If $\underline{y} + \underline{a} \geq \underline{a}$, then $(\underline{y} + \underline{a})_i \geq \underline{a}_i$ for each i in $I(1, n)$. This implies that $\underline{y}_i \geq 0$, and hence that $\underline{y} \in [I(0, \infty)]^n$, the definition of the latter space. The above argument is reversible, so that the converse also holds.

Theorem 2.3. If $\underline{a}$ and $\underline{b}$ are members of L , and $\underline{a} \leq \underline{b}$, then

$$L(\underline{b}) + (\underline{b} - \underline{a}) \subseteq L(\underline{a}).$$

Proof. If $\underline{t} \in L(\underline{b})$, then $\underline{t} + \underline{b} \geq \underline{b}$ and $\underline{t} + \underline{b} \in L$. This implies that $\underline{t} + \underline{b} \geq \underline{a}$, which in turn implies that $\underline{t} + \underline{b} - \underline{a} \in L(\underline{a})$, or that $L(\underline{b}) + (\underline{b} - \underline{a}) \subseteq L(\underline{a})$, from Definitions 1.5 and 1.7.

Repetitions in the structure of a general set L may be studied by means of an equivalence relation $\mathcal{S}$ defined between pairs of elements of L .

Definition 2.3. Let $\underline{a}$ and $\underline{b}$ be elements of L . The binary relation $\mathcal{S}$ is defined by $\underline{a} \mathcal{S} \underline{b}$ if and only if $L(\underline{a}) = L(\underline{b})$.

The fundamental property of the $\mathcal{S}$ -relation is that $\mathcal{S}$ is an equivalence relation, and partitions L into mutually exclusive classes.

Theorem 2.4. The relation $\mathcal{S}$ is an equivalence relation.

Proof. The $\mathcal{S}$ -relation is defined in terms of set equality. Since set equality is reflexive, symmetric and transitive, set equality and the $\mathcal{S}$ -relation are equivalence relations.

If it is known that $\underline{a} \mathcal{S} \underline{b}$, then $\underline{a}$ and $\underline{b}$ have the stated property that $L(\underline{a}) = L(\underline{b})$, but if $\underline{a} \leq \underline{b}$, several stronger results may be derived. These concern the $\mathcal{S}$ -equivalence classes, or simply $\mathcal{S}$ -classes of elements $\underline{c} \geq \underline{a}$.

Theorem 2.5. Let $\underline{a}$ and $\underline{b}$ be elements of L such that $\underline{a} \mathcal{S} \underline{b}$.

If $\underline{t} \in [I(0, \infty)]^n$ and $\underline{a} + \underline{t} \in L$, then $\underline{b} + \underline{t} \in L$ and $\underline{a} + \underline{t} \mathcal{S} \underline{b} + \underline{t}$.

Proof. Since $\underline{a} \mathcal{S} \underline{b}$, and $\underline{t} \in [I(0, \infty)]^n$, $\underline{a} + \underline{t}$ is a member of L if and only if $\underline{b} + \underline{t}$ is in L , and since $\underline{a} + \underline{t} \in L$ by hypothesis, $\underline{b} + \underline{t} \in L$. If $\underline{t}(1)$ is a member of $L(\underline{a} + \underline{t})$, then $\underline{t}(1) + \underline{t} + \underline{a} \in L$, and $\underline{t}(1) + \underline{t} \in L(\underline{a}) = L(\underline{b})$. Then $\underline{t}(1) + \underline{t} + \underline{b} \geq \underline{t} + \underline{b}$ and $\underline{t}(1) \in L(\underline{b} + \underline{t})$. This implies that

$L(\underline{a} + \underline{t}) \subseteq L(\underline{b} + \underline{t})$. Since the hypotheses are symmetric in $\underline{a}$ and $\underline{b}$, a similar argument shows that $L(\underline{b} + \underline{t}) \subseteq L(\underline{a} + \underline{t})$, hence the desired conclusion, $L(\underline{a} + \underline{t}) = L(\underline{b} + \underline{t})$, holds.

Corollary 2.2. Let $\underline{a}$ and $\underline{b}$ be elements of L such that $\underline{a} \mathcal{S} \underline{b}$.

If $\underline{t} \in [I(0, \infty)]^n$, then $\underline{a} \mathcal{S} \underline{a} + \underline{t}$ if and only if $\underline{b} \mathcal{S} \underline{b} + \underline{t}$.

Proof. From Theorem 2.5, the presence of either $\underline{a} + \underline{t}$ or $\underline{b} + \underline{t}$ in L implies the presence of the other, and the hypothesis that $\underline{a} \mathcal{S} \underline{a} + \underline{t}$ implies that $\underline{a} + \underline{t} \in L$. The second conclusion of Theorem 2.5 implies that $\underline{a} + \underline{t} \mathcal{S} \underline{b} + \underline{t}$, and the conclusion follows from the transitivity of the $\mathcal{S}$ relation.

The result of Theorem 2.5 may be rephrased and slightly weakened to put it in another usable form.

Theorem 2.6. If $\underline{a}$ and $\underline{b}$ are elements of L , such that $\underline{a} \leq \underline{b}$, and $\underline{a} \mathcal{S} \underline{a} + \underline{t}$ for some $\underline{t}$ in $[I(0, \infty)]^n$, then $\underline{b} \mathcal{S} \underline{b} + \underline{t}$.

Proof. If in Theorem 2.5, $\underline{b}$ is replaced by $\underline{a} + \underline{t}^*$, and $\underline{a} + \underline{t}$ by $\underline{b}^*$, then the conclusion is that $\underline{a} + \underline{t}^* + \underline{t} = \underline{b} + \underline{t} = \underline{b}^* + \underline{t}^* \in L$ and that $\underline{b}^* \mathcal{S} \underline{b}^* + \underline{t}^*$. When the asterisks are removed, this is the conclusion of the present theorem.

This theorem has an important corollary concerning the cycling properties of the set L .

Corollary 2.3. If $\underline{a}$ and $\underline{c}$ are elements of L , $\underline{a} \leq \underline{c}$ and $\underline{a} \mathcal{S} \underline{c}$, then $\underline{a} \mathcal{S} \underline{a} + m(\underline{c} - \underline{a})$ for any m in $I(0, \infty)$.

Proof. The corollary is proved by induction. If $m = 0$, the result holds by reflexivity of $\mathcal{S}$, and $m = 1$ implies that $\underline{a} \mathcal{S} \underline{a} + \underline{c} - \underline{a} = \underline{c}$, which

holds by hypothesis. The induction hypothesis is that if $\underline{a} \mathcal{S} \underline{c}$, then $\underline{a} + m(\underline{c} - \underline{a}) \mathcal{S} \underline{a}$ for all m in $I(0, k-1)$. Theorem 2.5 then implies that $\underline{c} = \underline{a} + (\underline{c} - \underline{a}) \mathcal{S} \underline{a} + (k-1)(\underline{c} - \underline{a}) + (\underline{c} - \underline{a}) = \underline{a} + k(\underline{c} - \underline{a})$. The transitivity of the $\mathcal{S}$ -relation then implies that $\underline{a} \mathcal{S} \underline{a} + k(\underline{c} - \underline{a})$, so that the theorem holds for all m in $I(0, k)$, and hence for all m in $I(0, \infty)$.

An immediate consequence of Theorem 2.1 and Corollary 2.3 is the following theorem, which concerns the order of $\underline{A}_i$.

Theorem 2.7. Let $\underline{A}_i$ be an $\mathcal{S}$ -class of L . If $\underline{A}_i$ is finite, then $R(\underline{A}_i) = \underline{A}_i$. If $R(\underline{A}_i) = \underline{A}_i$, and $\underline{A}_i \subseteq [I(c, \infty)]^n$, then $\underline{A}_i$ is finite.

Proof. If $\underline{A}_i$ is finite and $R(\underline{A}_i) \neq \underline{A}_i$, then there is an element $\underline{a}$ in $\underline{A}_i$ which is not in $R(\underline{A}_i)$. Since $\underline{a}$ is not an element of $R(\underline{A}_i)$, there exists an element $\underline{b}$ in $\underline{A}_i$ such that $\underline{b} < \underline{a}$, so that Corollary 2.3 implies that $\underline{b} + m(\underline{a} - \underline{b}) \mathcal{S} \underline{b}$ for all m in $I(0, \infty)$, and $\underline{A}_i$ is infinite, contrary to the hypothesis. Thus $R(\underline{A}_i) = \underline{A}_i$.

If $R(\underline{A}_i) = \underline{A}_i$ and $\underline{A}_i \subseteq [I(c, \infty)]^n$, then Theorem 2.1 implies that $R(\underline{A}_i)$ is finite, hence that $\underline{A}_i$ is also finite. This is the desired conclusion.

III. CYCLING PROPERTIES OF $\mathcal{S}$ -CLASSES

In the previous section, general properties of the $\mathcal{S}$ -relation were derived. In this section a finite set $Q(\underline{A}_i)$ is defined for each $\mathcal{S}$ -class $\underline{A}_i$ of L . The elements of $Q(\underline{A}_i)$ are denoted by $q(j, \underline{A}_i)$, as j ranges over $I(1, n(\underline{A}_i))$. The set $Q(\underline{A}_i)$ has the property that if $\underline{a} \in \underline{A}_i$, $\underline{b} \in L$ and $\underline{b} \geq \underline{a}$, then $\underline{a} \mathcal{S} \underline{b}$ if and only if there exist coefficients c_j in $I(0, \infty)$ for which $\underline{b} = \underline{a} + \sum_j c_j q(j, \underline{A}_i)$. The theorems of the section show that the definition of the set $Q(\underline{A}_i)$ is meaningful and establish necessary and sufficient conditions that the linear combinations be unique.

Definition 2.4. Let $\underline{A}_i$ be any $\mathcal{L}$ -class of L . If $\underline{a} \in \underline{A}_i$, then the set $\hat{L}(\underline{a})$ is defined by $\hat{L}(\underline{a}) = \{\underline{d} \mid \underline{d} + \underline{a} \in \underline{A}_i \text{ and } \underline{d} \in L(\underline{a})\}$. The set $\hat{L}(\underline{A}_i)$ is defined to be $\hat{L}(\underline{a})$ for any $\underline{a}$ in $\underline{A}_i$. The set $Q(\underline{A}_i)$ is defined by $Q(\underline{A}_i)$ is defined by

$$Q(\underline{A}_i) = \{q(j, \underline{A}_i) \mid q(j, \underline{A}_i) \in R(\hat{L}(\underline{A}_i) - \{(0)\})\}.$$

The definition of the sets $Q(\underline{A}_i)$ means that the elements $q(j, \underline{A}_i)$ are the smallest elements of $L(\underline{a})$, except for 0 , for which $\underline{a} + q(j, \underline{A}_i)$ is a member of $\underline{A}_i$ and of $\hat{L}(\underline{a})$. The following theorem shows that the extrapolation from $\hat{L}(\underline{a})$ to $\hat{L}(\underline{A}_i)$ is independent of $\underline{a}$, as long as $\underline{a}$ is in $\underline{A}_i$, and that $\hat{L}(\underline{A}_i)$ is the set of all non-negative integral combinations of the elements of $Q(\underline{A}_i)$.

Theorem 2.8. Let $\underline{A}_i$ be any $\mathcal{L}$ -class of L . Then $\hat{L}(\underline{a}) = \hat{L}(\underline{b})$ for any pair of elements $\underline{a}, \underline{b}$ in $\underline{A}_i$. The set $\hat{L}(\underline{A}_i)$ is equal to $\left\{ \sum_j c_j q(j, \underline{A}_i) \right\}$, where the coefficients c_j each range over $I(0, \infty)$, and $q(j, \underline{A}_i)$ is an element of $Q(\underline{A}_i)$. The set $Q(\underline{A}_i)$ is finite.

Proof. In order to prove the first assertion, let $\underline{a}$ and $\underline{b}$ be elements of $\underline{A}_i$ and let $\underline{d}$ be a member of $\hat{L}(\underline{a})$. Theorem 2.5 implies that $\underline{b} + \underline{d} \in L$ and that $\underline{b} \not\leq \underline{b} + \underline{d}$. This implies that $\underline{d} \in L(\underline{b})$ and that $\underline{d} \in \hat{L}(\underline{b})$, so that $\hat{L}(\underline{a}) \subseteq \hat{L}(\underline{b})$. Since the hypotheses are symmetric, the reverse inclusion holds and $\hat{L}(\underline{a}) = \hat{L}(\underline{b})$, for any $\underline{a}$ and $\underline{b}$ in $\underline{A}_i$.

From the definition of $\hat{L}(\underline{A}_i)$, for the second assertion it suffices to show that $\hat{L}(\underline{g}) = \left\{ \sum_j c_j q(j, \underline{A}_i) \right\}$ for some $\underline{g}$ in $\underline{A}_i$. If $q(k, \underline{A}_i) \in Q(\underline{A}_i)$, then $\underline{g} + q(k, \underline{A}_i)$ is an element of $\underline{A}_i$, so that by Corollary 2.3, $\underline{g} + c_k q(k, \underline{A}_i) \not\leq \underline{g}$. Theorem 2.6 then implies that $\underline{g} \not\leq \underline{g} + \sum_k c_k q(k, \underline{A}_i)$, where the summation index k ranges over $I(1, n(\underline{A}_i))$ and $n(\underline{A}_i)$ is the number of vectors $q(j, \underline{A}_i)$ in $Q(\underline{A}_i)$. This means that $\hat{L}(\underline{g}) \supseteq \left\{ \sum_j c_j q(j, \underline{A}_i) \right\}$ where $c_j \in I(0, \infty)$.

Next, let $\underline{d}$ be any element of $\hat{L}(\underline{g})$, and let $\underline{d}^* = \sum_{j \in I(1, n(\underline{A}_i))} c_j^* q(j, \underline{A}_i)$ satisfy $\underline{d}^* \leq \underline{d}$ and $\underline{d}^* + q(k, \underline{A}_i) \not\leq \underline{d}$ for each k in $I(1, n(\underline{A}_i))$.

Since $\underline{d}^* \in \hat{L}(\underline{g})$, $\underline{d} - \underline{d}^* \in \hat{L}(\underline{g} + \underline{d}^*) = L(\underline{g})$. If $\underline{d} - \underline{d}^* = 0$, then $\underline{d} = \underline{d}^*$ and $\underline{d}$ has the prescribed form. If $\underline{d} - \underline{d}^* > 0$, then $\underline{d} - \underline{d}^* \not\leq q(j, \underline{A}_i)$ for any j in $I(1, n(\underline{A}_i))$, which contradicts the hypothesis that $Q(\underline{A}_i)$ is the set of all non-zero minimal elements of $\hat{L}(\underline{A}_i)$. Thus the first alternative must hold, and $\hat{L}(\underline{g}) \subseteq \left\{ \sum_j d_j q(j, \underline{A}_i) \right\}$ and the stated conclusion holds.

The last assertion follows from the definition of $Q(\underline{A}_i)$ and Theorem 2.1.

The second assertion of Theorem 2.8 provides the motivation for introducing the set $Q(\underline{A}_i)$, since the assertion means that the finite set $Q(\underline{A}_i)$ yields complete information about the recurrence properties of the members of $\underline{A}_i$ as the component values increase. Theorem 2.9 shows the influence of ordering in L upon the sets $Q(\underline{A}_i)$.

Theorem 2.9. If $\underline{a}$ and $\underline{b}$ are elements of L , $\underline{a} \leq \underline{b}$ and $\underline{A}_k$ and $\underline{A}_m$ are the $\mathcal{S}$ -classes of $\underline{a}$ and $\underline{b}$ respectively, then for each i in $I(1, n(\underline{A}_k))$, $q(i, \underline{A}_k) = \sum_{j \in I(1, n(\underline{A}_m))} \mu(i, j; k, m) q(j, \underline{A}_m)$, where $\mu(i, j; k, m) \in I(0, \infty)$ for each i and j .

Proof. From Theorem 2.6, whenever $\underline{a} \mathcal{S} \underline{a} + \underline{t}$ and $\underline{a} \leq \underline{b}$, then $\underline{b} \mathcal{S} \underline{b} + \underline{t}$, where $\underline{t} \in [I(0, \infty)]^n$. Theorem 2.8 implies that for any $\underline{t}$ such that $\underline{b} \mathcal{S} \underline{b} + \underline{t}$, $\underline{t} = \sum_{j \in I(1, n(\underline{A}_m))} c_j q(j, \underline{A}_m)$. In particular such a representation holds if $\underline{t} = q(i, \underline{A}_k)$. This proves the theorem.

The final theorem of the section characterizes one form of a linear independence condition which may be imposed upon the elements of $Q(\underline{A}_i)$.

Theorem 2.10. Let $\underline{A}_i$ be an $\mathcal{S}$ -class of a set L . Then the elements of $Q(\underline{A}_i)$ have disjoint indices for their non-zero components if and

only if for any $\underline{a}$ in $\underline{A}_i$ and for any coefficient vectors $c(1)$ and $c(2)$ in $[I(0, \infty)]^{n(\underline{A}_i)}$,

$$[\underline{a} + \sum_{m \in I(1, n(\underline{A}_i))} c_m(1) q(m, \underline{A}_i)] \vee [\underline{a} + \sum_m c_m(2) q(m, \underline{A}_i)]$$

is in $\underline{A}_i$.

Proof. Let the $q(j, \underline{A}_i)$ have disjoint indices on their non-zero components, that is, $q(j, \underline{A}_i) \wedge q(k, \underline{A}_i) = \underline{0}$ if j and k are in $I(1, n(\underline{A}_i))$ and j is not equal to k . This equation implies that $\sum_j c_j q(j, \underline{A}_i) = \vee_j c_j q(j, \underline{A}_i)$, for any coefficient vector c in $[I(0, \infty)]^{n(\underline{A}_i)}$, by Lemma 1.3.

Let $\underline{a}$ be any element of $\underline{A}_i$ and let $c(1)$ and $c(2)$ be any coefficient vectors in $[I(0, \infty)]^{n(\underline{A}_i)}$. Then, as m ranges over $I(1, n(\underline{A}_i))$,

$$\begin{aligned} & [\underline{a} + \sum_m c_m(1) q(m, \underline{A}_i)] \vee [\underline{a} + \sum_m c_m(2) q(m, \underline{A}_i)] \\ &= [\underline{a} + \bigvee_m c_m(1) q(m, \underline{A}_i)] \vee [\underline{a} + \bigvee_m c_m(2) q(m, \underline{A}_i)] \\ &= \underline{a} + [\bigvee_m c_m(1) q(m, \underline{A}_i) \vee \bigvee_m c_m(2) q(m, \underline{A}_i)] \\ &= \underline{a} + [\bigvee_m (c_m(1) \vee c_m(2)) q(m, \underline{A}_i)] \\ &= \underline{a} + \bigvee_m c_m q(m, \underline{A}_i) \\ &= \underline{a} + \sum_m c_m q(m, \underline{A}_i), \quad \text{where } c_m = c_m(1) \vee c_m(2). \end{aligned}$$

The distributive law used in the second equation is a property of numerical vectors, from Lemma 1.2. The distributive rule employed in passing to the third equation follows from the assumption that the $q(m, \underline{A}_i)$ have disjoint non-zero component indices, which implies that each index spanned by $q(h, \underline{A}_i)$ has its value determined by the larger of the two components $c_h(1)$, $c_h(2)$ of the coefficient vector. Since each element of $c_m = c_m(1) \vee c_m(2)$ is in $I(0, \infty)$, $\underline{a} + \sum_m c_m q(m, \underline{A}_i)$ is a member of $\underline{A}_i$.

To prove the converse, let $[\underline{a} + \sum_m c_m(1) q(m, \underline{A}_i)] \vee [\underline{a} + \sum_m c_m(2) q(m, \underline{A}_i)]$ be an element of $\underline{A}_i$, for any $\underline{a}$ in $\underline{A}_i$ and any coefficient vectors $c(1)$, $c(2)$ in $[I(0, \infty)]^{n(\underline{A}_i)}$. In particular $\underline{a} + (q(m, \underline{A}_i) \vee q(k, \underline{A}_i))$ is in $\underline{A}_i$, for any indices m and k in $I(1, n(\underline{A}_i))$. From Theorem 2.8, $\underline{a} + q(m, \underline{A}_i) + q(k, \underline{A}_i)$ is in $\underline{A}_i$, and the inequality $q(m, \underline{A}_i) + q(k, \underline{A}_i) \geq q(m, \underline{A}_i) \vee q(k, \underline{A}_i) \geq q(m, \underline{A}_i)$ holds.

Each of the terms of the preceding inequality is a member of $\hat{L}(\underline{a})$, so that $q(m, \underline{A}_i) \vee q(k, \underline{A}_i) - q(m, \underline{A}_i)$ is a member of $L(\underline{a} + q(m, \underline{A}_i))$. Since $q(k, \underline{A}_i) \geq q(m, \underline{A}_i) \vee q(k, \underline{A}_i) - q(m, \underline{A}_i)$ and $q(k, \underline{A}_i)$ is a minimal element of $(\hat{L}(\underline{A}_i) - \{\underline{0}\})$, either the strict inequality does not hold, and $q(k, \underline{A}_i) = q(m, \underline{A}_i) \vee q(k, \underline{A}_i) - q(m, \underline{A}_i)$, or $q(m, \underline{A}_i) \vee q(k, \underline{A}_i) - q(m, \underline{A}_i) = 0$. The second possibility implies that $q(m, \underline{A}_i) = q(k, \underline{A}_i)$, which contradicts the hypothesis that $q(m, \underline{A}_i)$ is a minimal element of $\hat{L}(\underline{A}_i) - \{\underline{0}\}$. Thus $q(k, \underline{A}_i) + q(m, \underline{A}_i) = q(m, \underline{A}_i) \vee q(k, \underline{A}_i)$, which implies that $q(k, \underline{A}_i) \wedge q(m, \underline{A}_i) = 0$. This is the statement that $q(k, \underline{A}_i)$ and $q(m, \underline{A}_i)$ have disjoint indices for their non-zero components.

Example 1. In order to illustrate the concepts of this section, consider the following examples shown in Figure 1 on page 25. Let L consist of all vectors of the forms $c(1) \cdot (110) + c(2) \cdot (011)$ and $(100) + c(1) \cdot (110) + c(2) \cdot (011)$ as $c(1)$ and $c(2)$ range independently over $I(\infty, \infty)$. The set L has two $\mathcal{S}$ -classes corresponding to the two sets of vectors in the definition. To show this, let $\underline{a}$ and $\underline{b}$ be two elements of the form $c(1) \cdot (110) + c(2) \cdot (011)$, and let $\underline{d}$ be an element of $L(\underline{a})$. This implies that $\underline{d} = d(1) \cdot (110) + d(2) \cdot (011)$ or $\underline{d} = d(1) \cdot (110) + d(2) \cdot (011) + (100)$, where in the first case $d(1)$ and $d(2)$ are members of $I(0, \infty)$, and in the second case, $d(1)$ and $d(2)$ are both members of $I(0, \infty)$, or $d(1) = -1$ and $d(2) \in I(1, \infty)$. In either case $\underline{d} + \underline{b}$ is in L ,

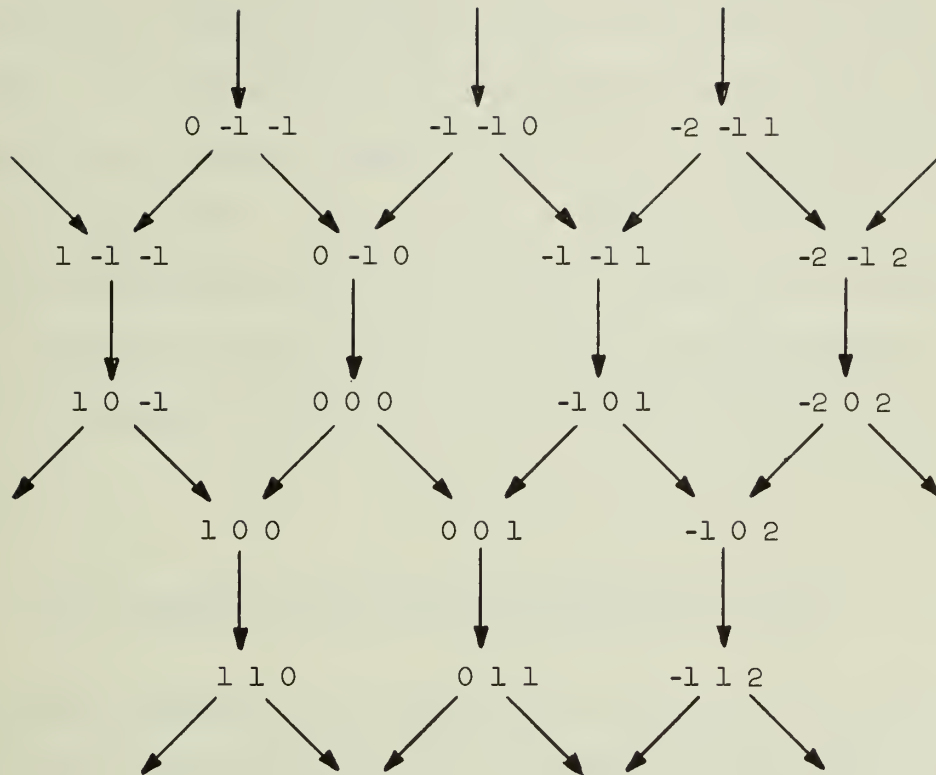


Figure 1

Example of L and Partition into $\mathcal{L}$ -Classes

from the definition of L , and $L(\underline{a}) \subseteq L(\underline{b})$. Since the hypotheses are symmetric, $L(\underline{a}) = L(\underline{b})$. If $\underline{a}$ and $\underline{b}$ are of the form $(100) + c(1) \cdot (110) + c(2) \cdot (011)$, then $\underline{d}$ in $L(\underline{a})$ means that $\underline{d} = d(1) \cdot (110) + d(2) \cdot (011)$ for $d(1)$ and $d(2)$ in $I(0, \infty)$, or $\underline{d} = d(1) \cdot (110) + d(2) \cdot (011) - (100)$ where $d(1)$ is in $I(1, \infty)$ and $d(2)$ is a member of $I(0, \infty)$. Again the vector $\underline{b} + \underline{d}$ is in L for any $\underline{d}$ in $L(\underline{a})$, so that $L(\underline{a}) \subseteq L(\underline{b})$, and from symmetry $L(\underline{b}) = L(\underline{a})$. From the description of the two sets of elements of $L(\underline{a})$, if $\underline{a} = c(1) \cdot (110) + c(2) \cdot (011)$ and $\underline{b} = d(1) \cdot (110) + d(2) \cdot (011) + (100)$, then $L(\underline{a}) \neq L(\underline{b})$.

The two $\mathcal{L}$ -classes have the same $Q(\underline{A}_1)$, namely $\{(110), (011)\}$. These two vectors do not have disjoint indices on their non-zero components, so that the union of two unordered elements of the same $\mathcal{L}$ -class does not lie in the class.

IV. PROPERTY II AND BASIS REQUIREMENTS FOR $Q(\underline{A}_1)$

In this section, a set of conditions is derived for a set L to have the property that for each $\mathcal{L}$ -class $\underline{A}_i$, the elements of $Q(\underline{A}_i)$ have disjoint indices for their non-zero components. These conditions include the restrictions that the number of distinct $\mathcal{L}$ -classes of L be finite and that L be closed under $\vee$.

The condition that L be the union of a finite number of $\mathcal{L}$ -classes is employed frequently enough to warrant a special terminology.

Definition 2.5. The set L is defined to have property II if

$$L = \bigcup_{i \in I(1, r)} \underline{A}_i, \text{ where } \underline{A}_i \text{ is an } \mathcal{L}\text{-class for each } i \text{ in } I(1, r), \text{ for some } r \text{ in } I(1, \infty).$$

The first theorem in the section shows the equivalence of the statements that there is some element of $\underline{A}_i$ greater than each of two given members $\underline{b}$ and $\underline{c}$ of

$\underline{a}_1$ and that $\underline{b} \vee \underline{c}$ is in $\underline{A}_1$, under the hypotheses that L is closed under $\vee$ and that L has property II.

Theorem 2.11. Let L have the properties that if $\underline{a}$ and $\underline{b}$ are elements of L , then $\underline{a} \vee \underline{b}$ is in L , and that L has property II. If $\underline{b}$ and $\underline{c}$ are elements of $\underline{A}_1$, an $\mathcal{A}$ -class of L , there exists an element $\underline{a}$ of $\underline{A}_1$ such that $\underline{a} \geq \underline{b}$ and $\underline{a} \geq \underline{c}$ if and only if $\underline{b} \vee \underline{c}$ is in $\underline{A}_1$.

Proof. $\Leftarrow$ If $\underline{b} \vee \underline{c} \in \underline{A}_1$, then $\underline{b} \vee \underline{c} \geq \underline{b}$ and $\underline{b} \vee \underline{c} \geq \underline{c}$, so that $\underline{b} \vee \underline{c}$ has the required properties.

$\Rightarrow$ If $\underline{b} \vee \underline{c} = \underline{a}$, there is nothing to prove. Otherwise, $\underline{b} \vee \underline{c} < \underline{a}$, since $\underline{a}$ is an upper bound for $\underline{b}$ and $\underline{c}$. If $\underline{r} \in L(\underline{b}) = L(\underline{c}) = L(\underline{a})$ then $\underline{r} + (\underline{b} \vee \underline{c}) \in L$, and $\underline{r} + (\underline{b} \vee \underline{c}) \geq \underline{b} \vee \underline{c}$, so that $\underline{r} \in L(\underline{b} \vee \underline{c})$. Thus $L(\underline{a}) \subseteq L(\underline{b} \vee \underline{c})$.

The reverse inclusion is demonstrated by contradiction. If $L(\underline{a}) \neq L(\underline{b} \vee \underline{c})$, there is some element $\underline{y}$ in $L(\underline{b} \vee \underline{c})$ such that $\underline{y} \notin L(\underline{a})$. The remainder of the proof shows that if this assertion holds, property II is violated.

For any p in $I(0, \infty)$, let $\underline{h}(p) = [\underline{b} \vee \underline{c} + (2\underline{a} - \underline{b} - \underline{c}) + (p-1)(\underline{a}-\underline{b})] \vee [\underline{b} \vee \underline{c} + (2\underline{a} - \underline{b} - \underline{c}) + (p-1)(\underline{a}-\underline{c})]$. By Lemma 1.2, $\underline{h}(p) = \underline{a} + p(\underline{a} - \underline{b} \vee \underline{c})$. Thus $\underline{h}(0) = \underline{a}$, $\underline{h}(1) = \underline{b} \vee \underline{c} + (2\underline{a} - \underline{b} - \underline{c})$, and each $\underline{h}(p)$ is an element of L . Let $\underline{y}(t) = \underline{y} + t[(2\underline{a} - \underline{b} - \underline{c}) - (\underline{a} - \underline{b} \wedge \underline{c})] = \underline{y} + t(\underline{a} - \underline{b} \vee \underline{c})$.

By hypothesis $\underline{y}(0) \in L(\underline{h}(1))$ and $\underline{y}(0) \notin L(\underline{h}(0))$. If $\underline{y}(t) \in L(\underline{h}(t+1))$, then $(2\underline{a} - \underline{b} - \underline{c}) - (\underline{a} - \underline{b} \wedge \underline{c}) = (\underline{a} - \underline{b} \vee \underline{c}) \geq 0$ and $\underline{y}(t) + (2\underline{a} - \underline{b} - \underline{c}) + \underline{h}(t+1) \in L$. This implies that $\underline{y}(t+1) + (\underline{a} - \underline{b} \vee \underline{c}) \in L(\underline{h}(t+2))$. Furthermore $\underline{y}(t) \in L(\underline{h}(p+1))$ for any $p \geq t$, since $[\underline{y}(t) + (p-t)(\underline{a}-\underline{b})] \vee [\underline{y}(t) + (p-t)(\underline{a}-\underline{c})] \in L(\underline{h}(t+1))$.

If $\underline{y}(t) \notin L(\underline{h}(t) + (2\underline{a} - \underline{b} - \underline{c})) = L(\underline{h}(t))$, then $\underline{y}(t) + \underline{b} \vee \underline{c} \not\leq \underline{h}(t)$. Thus $\underline{y}(t) + \underline{b} \vee \underline{c} + (\underline{a} - \underline{b} \wedge \underline{c}) \not\leq \underline{h}(t) + (\underline{a} - \underline{b} \vee \underline{c}) \leq \underline{h}(t) + (\underline{a} - \underline{b} \wedge \underline{c}) = \underline{h}(t+1)$

and $\underline{y}(t+1) \not\leq \underline{h}(t+1)$. It has been shown that $\underline{y}(t) \in L(\underline{h}(q+1))$ for $q \geq t$ and $\underline{y}(t) \notin L(\underline{h}(t))$. Thus there is an infinite sequence of distinct sets $L(\underline{h}(q))$. This contradicts the hypothesis that there is only a finite number of distinct $\mathcal{L}$ -classes, hence of sets of this type, so that for each element $\underline{y}$ in $L(\underline{b} \vee \underline{c})$, $\underline{y} \in L(\underline{a})$. Thus $L(\underline{a}) \supseteq L(\underline{b} \vee \underline{c})$, and $L(\underline{a}) = L(\underline{b} \vee \underline{c})$.

This theorem apparently places a strong restriction upon the $\mathcal{L}$ -class $\underline{A}_i$, in requiring that there be some element $\underline{a}$ in $\underline{A}_i$ for which $\underline{a} \geq \underline{b}$ and $\underline{a} \geq \underline{c}$. The following corollary shows that the restriction may be altered to require that $\underline{a}$ be an element of $\underline{A}_i$ which satisfies $\underline{a} \leq \underline{b}$ and $\underline{a} \leq \underline{c}$, but which rules out the converse of the original theorem.

Corollary 2.4. If the hypotheses of Theorem 2.10 are altered by requiring that there exist an element $\underline{d}$ in $\underline{A}_i$, $\underline{d} \leq \underline{b}$, $\underline{d} \leq \underline{c}$, for $\underline{b}$, $\underline{c}$ in $\underline{A}_i$, then the sufficiency condition holds.

Proof. If $\underline{d} \leq \underline{b}$, $\underline{d} \leq \underline{c}$, then $\underline{a} = \underline{d} + (\underline{b} - \underline{d}) + (\underline{c} - \underline{d}) = \underline{b} + \underline{c} - \underline{d}$ satisfies the original hypotheses, and, the conclusion holds.

Example 2. At this point, an example illustrates why a theorem such as 2.10 is necessary. The given set L is diagrammed in Figure 2, page 29. For any $c(1)$ and $c(2)$ in $I(3, \infty)$, $(c(1), c(2), 0)$ and $(c(1), c(2), 1)$ are members of L . The set L has $\mathcal{L}$ -classes $\{(000)\}$, $\{(100), (001)\}$, $\{(c(1), c(2), 0) \mid c(1), c(2) \in I(1, \infty) \text{ and } c(1) + c(2) \geq 2\}$, and $\{(c(1), c(2), 1) \mid c(1), c(2) \in I(1, \infty) \text{ and } c(1) + c(2) \geq 2\}$. In this example the $\mathcal{L}$ -class $\{(100), (001)\}$ has exactly two members, which shows that finite $\mathcal{L}$ -classes exist which have more than one element. Thus some additional hypothesis must be invoked.

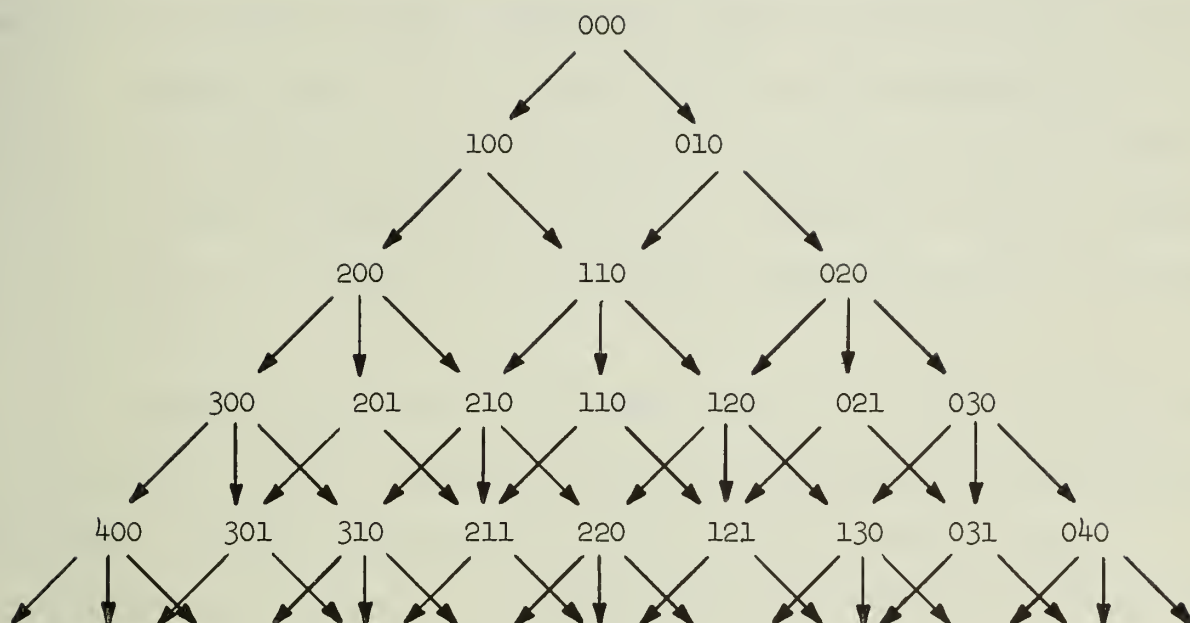


Figure 2

The Necessity for an Ordering Among Members of an $\mathcal{S}$ -Class

This example also shows that Theorem 2.10 is stronger than Corollary 2.4, since Theorem 2.10 implies that the union of two minimal elements, such as 200 and 110, is in $\underline{A}_i$ if, say, 320 is in $\underline{A}_i$, but the corollary does not yield this result.

The next theorem forms the connection between Theorems 2.9 and 2.10, concerning the basis or linear independence properties of the elements of $Q(\underline{A}_i)$.

Theorem 2.12. Let L be closed under $\vee$ and have property II. Let $\underline{a}$ be an element of $\underline{A}_i$, an $\mathcal{S}$ -class of L , and let $c(1), c(2)$ be coefficient vectors contained in $[I(0, \infty)]^{n(\underline{A}_i)}$. Let $\underline{a}(j) = \underline{a} + \sum_m c_m(j) q(m, \underline{A}_i)$ for $j = 1, 2$ where m is in $I(1, n(\underline{A}_i))$. Then $\underline{a}(1) \vee \underline{a}(2)$ is an element of $\underline{A}_i$.

Proof. Let $\underline{a}, c(1), c(2)$ be given, and let $d = c(1) + c(2)$, so that d is in $[I(0, \infty)]^{n(\underline{A}_i)}$. This implies that $\underline{a} + \sum_m d_m q(m, \underline{A}_i)$ is in $\underline{A}_i$, from the definition of $\hat{L}(\underline{A}_i)$ and Theorem 2.8. Since $d_m \geq c_m(1)$ and $d_m \geq c_m(2)$, $\underline{a} + \sum_m d_m q(m, \underline{A}_i) \geq \underline{a}(j)$ for $j = 1, 2$. Theorem 2.10 applied to the elements $\underline{a}(j)$ implies that $\underline{a}(1) \vee \underline{a}(2)$ is an element of $\underline{A}_i$.

Corollary 2.5. Let L be closed under $\vee$, and have property II. For any i in $I(1, r)$, the elements of $Q(\underline{A}_i)$ have disjoint indices on their non-zero components.

Proof. The hypothesis and conclusion of Theorem 2.11 form the hypothesis of Theorem 2.9, and the conclusion of Theorem 2.9 is the conclusion of this corollary.

Corollary 2.6. If L is closed under $\vee$ and has property II, then for any j, k , and $m, \mu(i, j; k, m)$ is non-zero for at most one i .

Proof. From Corollary 2.5, the elements of $Q(\underline{A}_{\underline{k}})$ have disjoint indices for their non-zero components, so that no member of $Q(\underline{A}_{\underline{k}})$ may be a summand of more than one $q(j, \underline{A}_{\underline{m}})$. This is the desired assertion.

In this section, the linear independence properties of the elements of the $Q(\underline{A}_{\underline{i}})$ have been obtained, based upon the hypotheses that L be closed under and have property II. Example 1 shows that property II alone does not suffice, even though L be a lattice and have the additional property that the covering relation in L implies the covering relation in $[I(-\infty, \infty)]^n$. Similarly, there exists an example of a lattice L which is closed under $\vee$, but in which an infinite number of $\mathcal{L}$ -classes appear and in which two elements $q(j, \underline{A}_{\underline{i}})$, $q(k, \underline{A}_{\underline{i}})$ of $Q(\underline{A}_{\underline{i}})$ have some common index h such that $q_h(j, \underline{A}_{\underline{i}}) \neq 0$ and $q_h(k, \underline{A}_{\underline{i}}) \neq 0$. Thus the theorems in their present form may not be weakened.

CHAPTER III

THE CONCEPT OF PERIODIC CLASS

In Chapter II, the concept of $\mathcal{S}$ -relation was introduced between pairs of elements of L , and the equivalence classes under $\mathcal{S}$ were shown to have properties related to the repetitive behavior of the elements of the set as the component values of the vectors increase. In this section certain subsets of $\mathcal{S}$ -classes are combined into still other classes, called periodic classes. The periodic class concept is applied in this section to obtain a bound on the amount of possible variation between two elements of the same periodic class when the value of any spanned component is fixed.

The concept of periodic class permits the introduction of a partial ordering $\mathcal{T}$ between periodic classes, such that the periodic classes form a lattice under the $\mathcal{T}$ -ordering.

I. THE PERIODIC CLASS

In this section, the periodic class is defined and its basic properties are developed.

Definition 3.1. If $\underline{d}$ is any element of L , let $\underline{A}(\underline{d})$ be defined to be the $\mathcal{S}$ -class which contains $\underline{d}$. The subset $\underline{B}$ of L is defined to be a periodic class of L if $\underline{B} = \{ \underline{a} \mid \underline{a} \in L \text{ and } M(\underline{A}(\underline{a})) = M(\underline{A}(\underline{b})) = M(\underline{B}) \text{ and } \underline{a} = \underline{b} \bmod N(\underline{B}) \}$. The set $Q(\underline{B})$ of periodicity vectors of $\underline{B}$ is defined to be $Q(\underline{A})$, where $\underline{A}$ is the $\mathcal{S}$ -class of any element of $\underline{B}$.

The definition of periodic class is quite inclusive, since a periodic class $\underline{B}$ contains all elements of L which have the same set of spanned component

indices and for which each non-spanned component has a constant value on all elements of $\underline{B}$. Theorem 3.1 and Corollary 3.1 show that all elements of the periodic class in fact have the same set of periodicity vectors.

Theorem 3.1. Let the elements $\underline{a}$ and $\underline{b}$ of L satisfy $\underline{a}_h = \underline{b}_h$ if h is not in $M(\underline{A}(\underline{a})) \cap M(\underline{A}(\underline{b}))$, where $\underline{A}(\underline{a})$ and $\underline{A}(\underline{b})$ are the $\mathcal{A}$ -classes of $\underline{a}$ and $\underline{b}$ respectively. Then there is a coefficient vector c in $[I(0, \infty)]^{n(\underline{A}(\underline{b}))}$ such that $\underline{a} \leq \underline{b} + \sum_j c_j q(j, \underline{A}(\underline{b}))$, where the range of j is $I(1, n(\underline{A}(\underline{b})))$. Also, $\hat{L}(\underline{a}) = \hat{L}(\underline{b})$.

Proof. The first step of the proof consists of constructing an element

$\underline{b}^* = \underline{b} + \sum_j c_j q(j, \underline{A}(\underline{b}))$ such that $\underline{b}^* \geq \underline{a}$. If h is an index such that $\underline{a}_h > \underline{b}_h$, then there is an integer m in $I(1, n(\underline{A}(\underline{b})))$ such that $q_h(m, \underline{A}(\underline{b})) > 0$, from the definition of $M(\underline{A}(\underline{b}))$. Thus there is an integer $\nu(m, h)$ in $I(1, \infty)$ such that $(\nu(m, h) - 1) q_h(m, \underline{A}(\underline{b})) < \underline{a}_h - \underline{b}_h \leq \nu(m, h) q_h(m, \underline{A}(\underline{b}))$, by the Euclidean algorithm. The vector $\underline{b}^*$ is defined by $\underline{b}^* = \underline{b} + \sum_m \nu(m) q(m, \underline{A}(\underline{b}))$, where $\nu(m) = \max\{\nu(m, h)\}$ and the maximum is taken over the set of h for which $q_h(m, \underline{A}(\underline{b})) \neq 0$ and $\underline{b}_h < \underline{a}_h$. Then $\underline{b}^*$ satisfies $\underline{b}^* \geq \underline{b} + \sum_m \sum_h \nu(m, h) q_h(m, \underline{A}(\underline{b})) \geq \underline{a}$, and $\underline{b}^*$ has the property asserted in the theorem.

This conclusion and Theorem 2.6 imply that $\hat{L}(\underline{b}^*) (= \hat{L}(\underline{b})) \supseteq \hat{L}(\underline{a})$.

Since the hypotheses of the theorem are symmetric, an element $\underline{a}^*$ may be found for which $\hat{L}(\underline{a}^*) (= \hat{L}(\underline{a})) \supseteq \hat{L}(\underline{b})$, so that $\hat{L}(\underline{a}) = \hat{L}(\underline{b})$, which is the conclusion of the theorem.

Corollary 3.1. If $\underline{a}$ and $\underline{b}$ are elements of $\underline{B}$, a periodic class of L , then $\hat{L}(\underline{a}) = \hat{L}(\underline{b})$, and $Q(\underline{a}) = Q(\underline{b})$.

Proof. Definition 3.1 implies that $\underline{a}$ and $\underline{b}$ satisfy the hypotheses of Theorem 3.1 and the second conclusion of that theorem is the first conclusion of the corollary.

The second assertion follows directly from the definition of $Q(\underline{a})$ as the set of minimal elements of $\hat{L}(\underline{a}) - \{\underline{0}\}$.

The precise relation between periodic classes and $\mathcal{S}$ -classes is spelled out in Theorem 3.2.

Theorem 3.2. If $\underline{A}$ is a finite $\mathcal{S}$ -class of L , then $\{\underline{a}\}$ is a periodic class for each $\underline{a}$ in $\underline{A}$. If $\underline{A}$ is infinite, and $\underline{a}$, an element of $\underline{A}$, is contained in a periodic class $\underline{B}$, then $\underline{b}$ in $\underline{A}$ satisfies $\underline{b} \in \underline{B}$ if and only if there is a coefficient vector c in $[I(-\infty, \infty)]^{n(\underline{A})}$ such that

$$\underline{b} = \underline{a} + \sum_{k \in I(1, n(\underline{A}))} c_k q(k, \underline{A}).$$

Proof. If $\underline{a}$ is an element of $\underline{A}$, a finite $\mathcal{S}$ -class, then no two elements of $\underline{A}$ are ordered, and $\hat{L}(\underline{a}) - \{\underline{0}\}$ and $Q(\underline{A})$ are vacuous. Thus any two elements $\underline{a}$ and $\underline{b}$ of $\underline{B}$ must satisfy $\underline{a}_h = \underline{b}_h$ for all h in $I(1, n)$, so that $\underline{a} = \underline{b}$. Thus the periodic class of $\underline{a}$ is $\{\underline{a}\}$.

Let $\underline{a}$ and $\underline{b}$ be elements of the infinite $\mathcal{S}$ -class $\underline{A}$, and of the periodic class $\underline{B}$. Theorem 3.1 implies that there is an element $\underline{b}^*$ in $\underline{A}$ of the form

$$\underline{b} + \sum_k c_k(1) q(k, \underline{A}) \text{ for which } \underline{b}^* \geq \underline{a}. \text{ Then } \underline{b}^* - \underline{a} \in \hat{L}(\underline{a}), \text{ and}$$

$$\underline{b}^* = \underline{a} + \sum_k c_k(2) q(k, \underline{A}), \text{ from Theorem 2.8. This implies that}$$

$$\underline{b} = \underline{a} + \sum_k (c_k(2) - c_k(1)) q(k, \underline{A}), \text{ where the range of } k \text{ is } I(1, n(\underline{A})). \text{ This shows the necessity of the second statement.}$$

If $\underline{a}$ and $\underline{b}$ are elements of $\underline{A}$ and $\underline{b} = \underline{a} + \sum_k c_k(k, \underline{A})$, then $\underline{a} = \underline{b} \bmod N(\underline{A})$, so that $\underline{a}$ and $\underline{b}$ are members of the same periodic class. This is the sufficiency condition of the second assertion.

A corollary to Theorem 3.2 relates the number of periodic classes to the number of $\mathcal{L}$ -classes, in case L has a lower bound. Such a relation is of interest when property II is invoked to obtain a linear independence property for $Q(\underline{B})$.

Corollary 3.2. If L has a lower bound, and the number of $\mathcal{L}$ -classes of L is finite, then the number of periodic classes of L is also finite.

Proof. From the definition of $Q(\underline{B})$ and of minimal element, each element $\underline{a}$ of $\underline{A}$, an $\mathcal{L}$ -class, lies in the same periodic class as at least one minimal element of $\underline{A}$. Thus the number of periodic classes which contain elements of $\underline{A}$ is less than or equal to the number of minimal elements of $\underline{A}$. If L is a subset of $[I(c, \infty)]^n$, each $\mathcal{L}$ -class has a finite number of minimal elements, and if the number of $\mathcal{L}$ -classes is also finite, the number of periodic classes of L is finite.

The converse of Corollary 3.2 is false, since if L is the set of pairs $(\underline{a}_1, \underline{a}_2)$, where $\underline{a}_1 \in I(0, \infty)$ and $\underline{a}_2(\underline{a}_1) \in I(0, \underline{a}_1)$, any set which contains all members of L for which $\underline{a}_1 - \underline{a}_2 = k$, where $k \in I(0, \infty)$ is characteristic of this set, is an $\mathcal{L}$ -class. Since the $\mathcal{L}$ -classes all have periodicity vector (11), all nodes are spanned and L has one periodic class. However, the number of $\mathcal{L}$ -classes is countably infinite.

II. THE STRUCTURE OF A PERIODIC CLASS AND THE FUNCTION λ

The second important property of the concept of periodic class is that the concept provides a framework for specifying the sequences of elements of L which are present as the various components increase. Theorem 2.8, which characterizes $\hat{L}(\underline{a})$ as the set of all non-negative integral combinations of the

$q(j, \underline{A})$, implies that much of the information about the sequences of elements of L may be obtained by considering the sets of component indices spanned by the elements $q(j, \underline{B})$ separately for each j . Such a program implies that the $q(j, \underline{B})$ have disjoint indices on their non-zero components. In order to achieve this condition, Corollary 2.5 states that it is sufficient to require that L be closed under $\vee$ and have property II.

Definition 3.2. Let $\underline{B}$ be a periodic class of L , and let $\underline{a}$ be any member of $\underline{B}$. Then the structure $Y(\underline{a}, q(m, \underline{B}))$ of $q(m, \underline{B})$ with respect to $\underline{a}$ is defined by $Y(\underline{a}, q(m, \underline{B})) = \{ \underline{b} \mid \underline{b} \in L(\underline{a}), \underline{b} = \underline{0} \bmod N(q(m, \underline{B})), \text{ and } \underline{b} \notin q(m, \underline{B}) \}$.

Theorems 3.3 through 3.6 display some invariance properties of $Y(\underline{a}, q(m, \underline{B}))$. Theorem 3.3 shows that the structure $Y(\underline{a}, q(m, \underline{B}))$ is dependent only upon the $\mathcal{L}$ -class of $\underline{a}$. The motivation for introducing the structure within the framework of the periodic class is that for any element $\underline{y}$ in $Y(\underline{a}, q(m, \underline{B}))$, $\underline{y} + \underline{a} \in \underline{B}$, where $\underline{B}$ is the periodic class which contains $\underline{a}$. Theorems 3.4 and 3.5 yield additional reasons for introducing the structure at this time. Theorem 3.4 shows that if $\underline{y}$ is any member of $Y(\underline{a}, q(m, \underline{B}))$ with $\underline{a}$ in $\underline{B}$, then $\underline{a} + \underline{y}$ is in $\underline{B}$. If L is closed under $\vee$ and possesses property II, then as long as $\underline{a} \in \underline{B}$, $Y(\underline{a}, q(m, \underline{B}))$ is dependent only upon the values of the components $\underline{a}_h$ of $\underline{a}$ for which $q_j(m, \underline{B}) \neq 0$. The conclusion of Theorem 3.6 states that the difference between two elements $\underline{a}, \underline{b}$ of a periodic class $\underline{B}$ is a sum of integral multiples of the periodicity vectors of $Q(\underline{B})$ and one member of the structure of each periodicity vector with respect to $\underline{a}$.

Theorem 3.3. Let $\underline{a}$ and $\underline{b}$ be elements of L which satisfy $\underline{a} \mathcal{L} \underline{b}$. If $\underline{A}$ is the $\mathcal{L}$ -class of $\underline{a}$ and $\underline{b}$, and m is in $I(1, n(\underline{A}))$, then $Y(\underline{a}, q(m, \underline{A})) = Y(\underline{b}, q(m, \underline{A}))$.

Proof. The statement that $\underline{a} \mathcal{L} \underline{b}$ implies that $L(\underline{a}) = L(\underline{b})$. Since the definition of $Y(\underline{a}, q(m, \underline{B}))$ depends only upon $q(m, \underline{B})$ ($= q(m, \underline{A})$) and upon $L(\underline{a})$, the theorem follows.

Theorem 3.4. Let $\underline{a}$ be an element of $\underline{B}$, where $\underline{B}$ is a periodic class of L . If $\underline{y} \in Y(\underline{a}, q(m, \underline{B}))$, then $\underline{a} + \underline{y} \in \underline{B}$.

Proof. It suffices to show that the $\mathcal{L}$ -classes $\underline{A}(\underline{a} + \underline{y})$ and $\underline{A}(\underline{a})$ of $\underline{a} + \underline{y}$ and $\underline{a}$ respectively, satisfy $M(\underline{A}(\underline{a} + \underline{y})) = M(\underline{A}(\underline{a}))$, and $\underline{a} = \underline{a} + \underline{y} \bmod N(\underline{A}(\underline{a}))$. The second condition is satisfied from the hypothesis that $\underline{y} \in Y(\underline{a}, q(m, \underline{B}))$. Theorem 3.1 then implies that $Q(\underline{A}(\underline{a} + \underline{y})) = Q(\underline{A}(\underline{a}))$, which implies that the first condition holds. Thus $\underline{a} + \underline{y} \in \underline{B}$.

Theorem 3.5. Let L be closed under $\vee$ and have property II. Let $\underline{B}$ be a periodic class of L , and let $\underline{a}$ and $\underline{b}$ be two elements of $\underline{B}$ such that for some m in $I(1, n(\underline{B}))$, $\underline{a}_h = \underline{b}_h$ for all h in $M(q(m, \underline{B}))$. Then $Y(\underline{a}, q(m, \underline{B})) = Y(\underline{b}, q(m, \underline{B}))$.

Proof. Without loss of generality, let $\underline{a}$ and $\underline{b}$ satisfy $\underline{a} \leq \underline{b}$, since otherwise, Theorem 3.1 implies that there exists an element $\underline{b}^*$ equal to $\underline{b} + \sum_{j \in I(1, n(\underline{B}))} c_j q(j, \underline{B})$ such that $\underline{b}^* \geq \underline{a}$. Since $\underline{b} \mathcal{L} \underline{b}^*$, Theorem 3.3 implies that $Y(\underline{b}, q(m, \underline{B})) = Y(\underline{b}^*, q(m, \underline{B}))$. Corollary 2.5 implies that the elements of $Q(\underline{B})$ have disjoint indices for their non-zero components, which in turn, implies that, in the definition of $\underline{b}^*$, $\underline{b}_h = \underline{b}_h^* = \underline{a}_h$.

Let $\underline{y}$ be an element of $Y(\underline{a}, q(m, \underline{B}))$. From the hypothesis that L is closed under $\vee$, $(\underline{a} + \underline{y}) \vee \underline{b} \in L$. Since $\underline{b} \geq \underline{a}$ and $\underline{b}_h \geq \underline{a}_h$ if $\underline{y}_h > 0$, the element $(\underline{a} + \underline{y}) \vee \underline{b}$ is equal to $\underline{b} + \underline{y}$, so that $\underline{y} \in L(\underline{b})$. This implies that $\underline{y} \in Y(\underline{b}, q(m, \underline{B}))$, from the assumption that $\underline{y} \in Y(\underline{a}, q(m, \underline{B}))$. Thus $Y(\underline{a}, q(m, \underline{B})) \subseteq Y(\underline{b}, q(m, \underline{B}))$. Since the hypotheses are symmetric, $Y(\underline{a}, q(m, \underline{B})) = Y(\underline{b}, q(m, \underline{B}))$.

Theorem 3.6. Let $\underline{B}$ be a periodic class of L , where L is closed under $\vee$ and has property II. If $\underline{a}$ and $\underline{b}$ are elements of $\underline{B}$, then for each i in $I(1, n(\underline{B}))$, there exists a unique integer k_i in $I(-\infty, \infty)$ and a unique element $\underline{z}_i$ in $Y(\underline{a}, q(i, \underline{B}))$ such that

$\underline{b} = \underline{a} + \sum_i k_i q(i, \underline{B}) + \sum_i \underline{z}(i)$. If $\underline{a}$ is in $\underline{B}$ and $\underline{b}$ is an element of $[I(-\infty, \infty)]^n$ such that $\underline{b} \geq \underline{a}$ and $\underline{b} = \underline{a} + \sum_i k_i q(i, \underline{B}) + \sum_i \underline{z}(i)$ for some set of coefficients k_i and elements $\underline{z}(i)$ of $Y(\underline{a}, q(i, \underline{B}))$, then

$\underline{b} \in \underline{B}$.

Proof. If $\underline{a}$ and $\underline{b}$ are two elements of $\underline{B}$, then, by Theorem 3.1 there exists an element $\underline{b}^*$ in $\underline{B}$ such that $\underline{b}^* \geq \underline{a}$ and $\underline{b}^* = \underline{b} + \sum_i d(i) q(i, \underline{B})$. Since the hypotheses of Corollary 2.5 hold, the elements of $Q(\underline{B})$ have disjoint indices on their non-zero components, and there exists a maximal element

$\underline{a}^* = \underline{a} + \sum_i c(i) q(i, \underline{B}) \leq \underline{b}^*$ for which $\underline{a} + q(j, \underline{B}) + \sum_i c(i) q(i, \underline{B}) \not\leq \underline{b}^*$ for any j in $I(1, n(\underline{B}))$.

To complete the proof of the assertion, it suffices to show that

$\underline{b}^* - \underline{a}^* = \sum_i \underline{z}(i)$, where $\underline{z}(i) \in Y(\underline{a}, q(i, \underline{B})) = Y(\underline{a}^*, q(i, \underline{B}))$. From the proof of Theorem 3.1, for each j in $I(1, n(\underline{B}))$ there exists an integer $f(j)$ such that $f(j) \cdot q_k(j, \underline{B}) = (\underline{b}^* - \underline{a}^*)_k$, for each k in $M(q(j, \underline{B}))$. Then

$(\underline{a}^* + \sum_{j \neq i} f(j) q(j, \underline{B})) \vee \underline{b}^*$ is in L , by closure under $\vee$, and

$(\underline{a}^* + \sum_{j \neq i} f(j) q(j, \underline{B})) \vee \underline{b}^* - (\underline{a}^* + \sum_{j \neq i} f(j) q(j, \underline{B}))$ is an element of

$Y(\underline{a}^* + \sum_{j \neq i} f(j) q(j, \underline{B}), q(i, \underline{B}))$, is equal to $(\underline{b}^* - \underline{a}^*)_i$ when $q_i(i, \underline{B}) \neq 0$, and equals 0 otherwise. The sum of such vectors as i ranges over $I(1, n(\underline{B}))$ is

$(\underline{b}^* - \underline{a}^*)$.

If $\underline{a} + \sum_i k_i q(i, \underline{B}) + \sum_i \underline{z}(i)$ and $\underline{a} + \sum_i k_i^* q(i, \underline{B}) + \sum_i \underline{z}^*(i)$ are distinct representations of some element $\underline{b}$ of $\underline{B}$, then

$\sum_i k_i q(i, \underline{B}) + \sum_i \underline{z}(i) = \sum_i k_i^* q(i, \underline{B}) + \sum_i \underline{z}^*(i)$ and

$\sum_i (k_i - k_i^*) q(i, \underline{B}) + \sum_i \underline{z}(i) = \sum_i \underline{z}(i)$. Since the hypotheses imply that $M(q(i, \underline{B})) \cap M(q(j, \underline{B})) = \emptyset$ whenever $i \neq j$, $(k_i - k_i^*) q(i, \underline{B}) + \underline{z}(i) = \underline{z}^*(i)$ for each i in $I(1, n(\underline{B}))$. If $k_i > k_i^*$ then $\underline{z}(i) \geq q(i, \underline{B})$ and not in $Y(\underline{a}, q(i, \underline{B}))$. Since the reversal of the inequality also contradicts the assumption that $\underline{z}(i) \in Y(\underline{a}, q(i, \underline{B}))$, $k_i = k_i^*$ and $\underline{z}(i) = \underline{z}^*(i)$ so that the representation of $\underline{b}$ is unique.

To prove the converse, let $\underline{a}$ be an element of $\underline{B}$ and let $\underline{b}$ be an element of $[I(-\infty, \infty)]^n$ for which $\underline{b} \geq \underline{a}$ and $\underline{b} = \underline{a} + \sum_i k(i)q(i, \underline{B}) + \sum_i \underline{z}(i)$, where $\underline{z}(i) \in Y(\underline{a}, q(i, \underline{B}))$. Since $\underline{a} \in \underline{B}$, $\underline{a}(0) = \underline{a} + \sum_i k(i)q(i, \underline{B}) \in \underline{B}$ by Theorem 3.2, and $\underline{z}(j) \in Y(\underline{a}(0), q(j, \underline{B}))$ for each j in $I(1, n(\underline{B}))$. Theorem 3.4 implies that $\underline{a}(0) + \underline{z}(1) = \underline{a}(1) \in \underline{B}$. For the induction hypothesis, let $\underline{z}(j)$ be an element of $Y(\underline{a}(i-1), q(j, \underline{B}))$, for j in $I(i, n(\underline{B}))$. By Theorem 3.2, $\underline{a}(i-1) + \underline{z}(i) \in \underline{B}$, and $\underline{z}(j) \in Y(\underline{a}(i), q(j, \underline{B}))$ for j in $I(i+1, n(\underline{B}))$, so that $\underline{a}(n) = \underline{a}(0) + \sum_i \underline{z}(i) = \underline{a} + \sum_i k(i)q(i, \underline{B}) + \sum_i \underline{z}(i) \in \underline{B}$. This is the desired conclusion.

Theorem 3.7 displays the relationship between $Y(\underline{b}, q(i, \underline{B}))$ and $Y(\underline{a}, q(i, \underline{B}))$, when $\underline{a}$ and $\underline{b}$ are both elements of the periodic class $\underline{B}$. An extension of the concept of equality mod $Q(\underline{B})$ is useful in stating this theorem.

Definition 3.3. If $\underline{a} \in [I(-\infty, \infty)]^n$ and $q \in [I(0, \infty)]^n$, then

$\text{Res } \underline{a} \text{ mod } q$ is defined to be $\underline{a} + c q$, where $c \in I(-\infty, \infty)$

defined by $(\underline{a} + c q)_k \geq 0$ for each k in $M(q)$ and $(\underline{a} + (c-1) q)_h < 0$ for some h in $M(q)$.

Thus if $q = (043)$, $\text{Res}(345) \text{ mod } q$ is (302) and $\text{Res}(-3-4-5)$ is (-341) .

Theorem 3.7. Let $\underline{B}$ be a periodic class of L , where L is closed under $\vee$ and has property II. Let $\underline{a}$ and $\underline{b}$ be elements of $\underline{B}$. Then $Y(\underline{b}, q(i, \underline{B})) = \{ \underline{z} \mid \underline{z} = \text{Res}((\underline{a} - \underline{b})^* + \underline{x}) \bmod q(i, \underline{B}) \text{ where } \underline{x} \in Y(\underline{a}, q(i, \underline{B})), \text{ and where } (\underline{a} - \underline{b})_h^* = 0 \text{ if } h \notin M(q(i, \underline{B})), \text{ and } (\underline{a} - \underline{b})_h^* = (\underline{a} - \underline{b})_h \text{ if } h \in M(q(i, \underline{B})) \}$.

Proof. By Theorems 3.5 and 3.6, it suffices to consider only elements

$\underline{a}$ and $\underline{b}$ for which $\underline{a} = \underline{b} \bmod N(q(i, \underline{B}))$, so that $(\underline{b} - \underline{a})^* = \underline{b} - \underline{a}$.

If $\underline{z} \in Y(\underline{b}, q(i, \underline{B}))$, then $\underline{b} + \underline{z} \in L$ and $\underline{b} + \underline{z} = \underline{a} + cq(i, \underline{B}) + \underline{z}(i) + \underline{z}$.

This implies that $\underline{z} = \underline{a} - \underline{b} + cq(i, \underline{B}) + \underline{z}(i) + \underline{z} = \underline{a} - \underline{b} + cq(i, \underline{B}) + \underline{z}^*$, where

$\underline{z}^* = \underline{z}(i) + \underline{z}$. The assumption that $\underline{z} \in Y(\underline{b}, q(i, \underline{B}))$ implies that

$\underline{z} = \text{Res}(\underline{a} - \underline{b} + \underline{z}^*) \bmod q(i, \underline{B})$. If $\underline{z} \notin Y(\underline{a}, q(i, \underline{B}))$, then $\underline{z}^* \not\equiv 2q(i, \underline{B})$ so that

$\underline{z}^* - q(i, \underline{B}) \not\equiv q(i, \underline{B})$ and $\underline{z}^* - q(i, \underline{B}) \in Y(\underline{a}, q(i, \underline{B}))$. Since the desired formula-

tion holds when $\underline{z}^* \in Y(\underline{a}, q(i, \underline{B}))$, the inclusion $\subseteq$ follows.

If $\underline{x} \in Y(\underline{a}, q(i, \underline{B}))$, then $\underline{x} + \underline{a} \in L$. From Theorem 3.6,

$\underline{x} + \underline{a} = \underline{x} + \underline{b} + cq(i, \underline{B}) + \underline{z}^*(i)$ and $\underline{x} + \underline{a} - \underline{b} = \underline{x} + \underline{z}^*(i) + cq(i, \underline{B})$. Upon

taking residues of both sides of the equation, an element $\underline{z}$ of $Y(\underline{b}, q(i, \underline{B}))$

is determined. This demonstrates that the relation $\supseteq$ holds, which completes

the proof.

Under the hypotheses that L has property II and is closed under $\vee$,

the five preceding theorems re-express the invariance of $L(\underline{a})$ under transla-

tions by integral multiples of the periodicity vectors. These results also

show that $\{\underline{a} + L(\underline{a})\} \cap \underline{A}$ is identical to the set of all non-negative trans-

lations of $Y(\underline{a}, q(i, \underline{B}))$ for each i in $I(1, n(\underline{B}))$.

Lemma 3.1 establishes an important property of the structure

$Y(\underline{a}, q(m, \underline{B}))$. This property, that no $\mathcal{J}$ -class is represented more than once in

the structure, is used in the proof of Theorem 3.8.

Lemma 3.1. Let L be closed under $\vee$ and have property II, and let $\underline{B}$ be a periodic class of L . If $\underline{a} \in \underline{B}$ and $m \in I(1, n(\underline{B}))$, and if $\underline{x}$ and $\underline{y}$ are elements of $Y(\underline{a}, q(m, \underline{B}))$ for which $\underline{a} + \underline{x} \not\leq \underline{a} + \underline{y}$, then $\underline{x} = \underline{y}$.

Proof. Let $\underline{x}$ and $\underline{y}$ be elements of $Y(\underline{a}, q(m, \underline{B}))$ such that $\underline{a} + \underline{x} \not\leq \underline{a} + \underline{y}$.

Theorems 3.4 and 3.2 imply that $\underline{a} + \underline{x} = \underline{a} + \underline{y} + \sum_{j \in I(1, n(\underline{B}))} c(j)q(j, \underline{B})$. The independence assumption implies that no two members $q(j, \underline{B})$ and $q(m, \underline{B})$ of $Q(\underline{B})$ possess some common index h such that $q_h(j, \underline{B}) \wedge q_h(m, \underline{B}) \neq 0$. This statement and the condition that $\underline{x} = \underline{y} = \underline{0} \pmod{N(q(m, \underline{B}))}$ imply that $\underline{x} = \underline{y} + c(m)q(m, \underline{B})$. From the hypothesis that $\underline{x}$ and $\underline{y}$ are elements of $Y(\underline{a}, q(m, \underline{B}))$, $\underline{x} \not\leq q(m, \underline{B})$ and $\underline{y} \not\leq q(m, \underline{B})$ which imply that $c(m) = 0$ and $\underline{x} = \underline{y}$.

The principal result of the section is that there exists a function $\lambda(q(m, \underline{B}), \underline{B})$ which establishes an upper bound for the members of $Y(\underline{a}, q(m, \underline{B}))$, and that the bound is independent of $\underline{a}$.

Definition 3.4. The function $\lambda(\underline{a}, q(m, \underline{B}))$ is defined to be the least element of $I(1, \infty)$ which satisfies $\underline{y} \leq \lambda(\underline{a}, q(m, \underline{B})) q(m, \underline{B})$ for all $\underline{y}$ in $Y(\underline{a}, q(m, \underline{B}))$. The function $\lambda(q(m, \underline{B}), \underline{B})$ is defined by $\lambda(q(m, \underline{B}), \underline{B}) = \max \lambda(\underline{a} + \underline{b}, q(m, \underline{B}))$, where the maximum is taken over all $\underline{b}$ in $Y(\underline{a}, q(m, \underline{B}))$.

Theorem 3.8. Let L have property II, be closed under $\vee$, and let $\underline{B}$ be a periodic class of L . Let $\underline{a}$ be any member of $\underline{B}$ and let m be any element of $I(1, n(\underline{B}))$. If $\underline{y}$ is any element of $Y(\underline{a}, q(m, \underline{B}))$ then $\underline{0} \leq \underline{y} \leq \lambda(q(m, \underline{B}), \underline{B}) q(m, \underline{B})$.

Proof. Lemma 3.1 and the hypothesis that L has property II and is closed under $\vee$ imply that the order $Y(\underline{a}, q(m, \underline{B}))$ is finite. For each $\underline{y}$ in $Y(\underline{a}, q(m, \underline{B}))$, $\underline{y} = \underline{0} \pmod{N(q(m, \underline{B}))}$, so that $\underline{y} < (1 + \max_{m \in I(1, n)} \underline{y}_k) q(m, \underline{B})$.

From the well-ordering properties of the integers, the set of integers $\{h(\underline{y})\}$ for which $\underline{y} < h(\underline{y})q(m, \underline{B})$ has a minimum, $\lambda(\underline{y}, \underline{a}, q(m, \underline{B}))$. Taking the least upper bound over the finite set of elements $\underline{y}$ in $Y(\underline{a}, q(m, \underline{B}))$, there exists an integer $\lambda(\underline{a}, q(m, \underline{B}))$ such that for any $\underline{y}$ in $Y(\underline{a}, q(m, \underline{B}))$, $0 \leq \underline{y} \leq \lambda(\underline{a}, q(m, \underline{B}))q(m, \underline{B})$.

Theorem 3.5 states that $Y(\underline{a}, q(m, \underline{B}))$ is independent of the components of $\underline{a}$ whose indices are contained in $M(q(j, \underline{B}))$ for $j \neq m$, so that all $Y(\underline{b}, q(m, \underline{B}))$ are obtained as $\underline{b}$ ranges over $\underline{a} + \underline{y}$ for $\underline{y}$ in $Y(\underline{a}, q(m, \underline{B}))$. Again the finiteness of $Y(\underline{a}, q(m, \underline{B}))$ implies that the least upper bound $\lambda(q(m, \underline{B}), \underline{B})$, over $\underline{y}$ in $Y(\underline{a}, q(m, \underline{B}))$, of $\lambda(\underline{a} + \underline{y}, q(m, \underline{B}))$ is defined, and that $\underline{y} \leq \lambda(q(m, \underline{B}), \underline{B})q(m, \underline{B})$. Since $0 \leq \underline{y}$ from the definition of $\underline{y}$, the theorem holds.

In the next chapter, Theorem 3.8 will be used to show that if two elements of a periodic class of L differ by more than $\lambda(q(m, \underline{B}), \underline{B})q(m, \underline{B})$ on those components whose indices are contained in $M(q(m, \underline{B}))$ then the two elements cannot correspond to $\underline{a} + \underline{x}$ and $\underline{a} + \underline{y}$, where $\underline{x}$ and $\underline{y}$ are elements of $Y(\underline{a}, q(m, \underline{B}))$ for any $\underline{a}$ in $\underline{B}$.

Example 3. To illustrate the concepts of this section, let L consist of the infinite set of vectors, a portion of the graph of which is shown in Figure 3, on page 43. The entire set L is a single periodic class containing the 6 $\mathcal{L}$ -classes, $\{000 + k \cdot (111)\}$, $\{100 + k \cdot (111)\}$, $\{110 + k \cdot (111)\}$, $\{001 + k \cdot (111)\}$, $\{101 + k \cdot (111)\}$ and $\{102 + k \cdot (111)\}$, where $k \in I(0, \infty)$. The structure of $q(L) = (111)$ with respect to 000 is contained within the dotted area of Figure 3, while $102 + Y(102, q(L))$ is contained within the dashed rectangle. From the figure, $\underline{y} \leq 2 \cdot (111)$ for each $\underline{y}$ in $Y(000, q(L))$, but $\underline{z} \not\leq 3 \cdot (111)$ for each $\underline{z}$ in $Y(102, q(L))$. Thus $231 \in Y(102, q(L))$ and $231 \not\leq 2 \cdot (111)$.

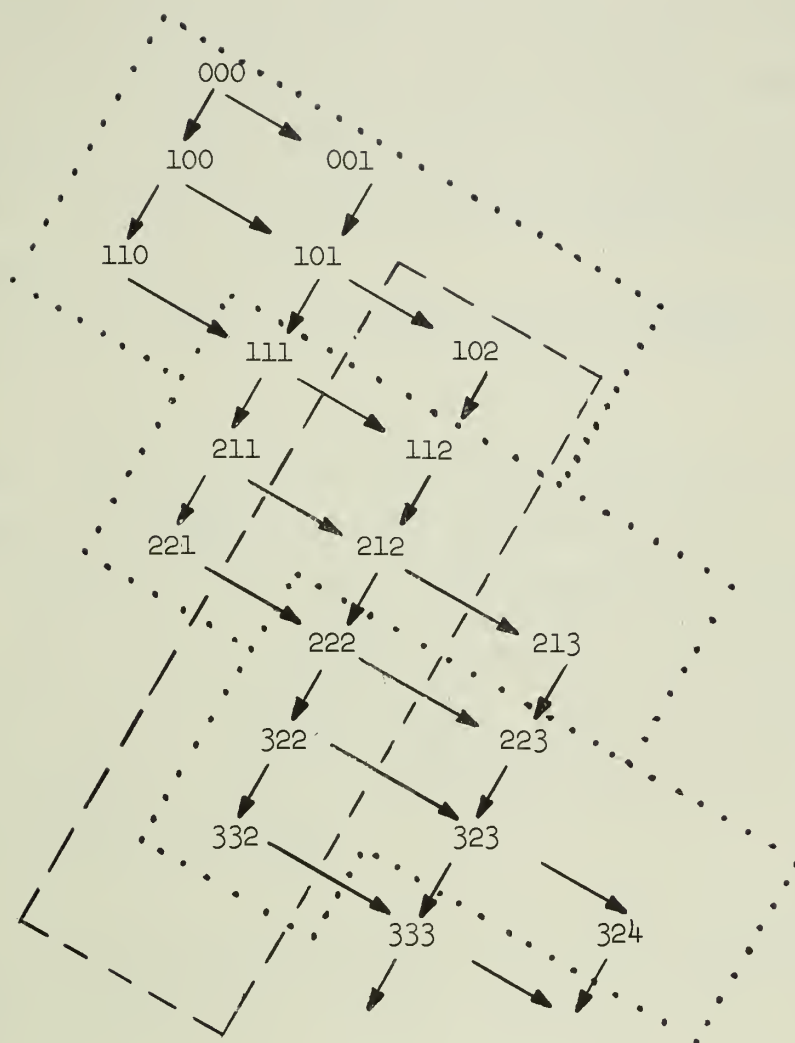


Figure 3

Example of Structure $Y(000, q(L))$ and $\lambda(q(L), L)$

III. THE $\mathcal{F}$ RELATION

In this section, the partial ordering $\leq$ between pairs of elements of L is extended to a partial ordering $\mathcal{F}$ between pairs of periodic classes of L .

Definition 3.5. Let $\underline{B}(i)$ and $\underline{B}(j)$ be periodic classes of L . Then $\underline{B}(i) \mathcal{F} \underline{B}(j)$ is defined to mean that for some element $\underline{a}$ in $\underline{B}(i)$ and some element $\underline{b}$ of $\underline{B}(j)$, $\underline{a} \leq \underline{b}$.

In several of the proofs, the stronger condition of Lemma 3.2 is employed.

Lemma 3.2. If $\underline{B}(i) \mathcal{F} \underline{B}(j)$ in L , and if $\underline{a}^*$ is any element of $\underline{B}(i)$, then there is an element $\underline{b}^*$ in $\underline{B}(j)$ for which $\underline{a}^* \leq \underline{b}^*$.

Proof. If $\underline{B}(i) \mathcal{F} \underline{B}(j)$, then there exist elements $\underline{a}$, $\underline{b}$ of $\underline{B}(i)$ and $\underline{B}(j)$ respectively, such that $\underline{a} \leq \underline{b}$. Theorem 3.1 implies that there is an element $\hat{\underline{a}} \geq \underline{a}$ in $\underline{B}(i)$ such that $\underline{a} \not\leq \hat{\underline{a}}$ and $\underline{a}^* \leq \hat{\underline{a}}$. Theorem 2.6 implies that there exists an element $\hat{\underline{b}}$ for which $\hat{\underline{b}} - \underline{b} = \hat{\underline{a}} - \underline{a}$, so that $\hat{\underline{a}} - \underline{a} + \underline{b} = \hat{\underline{b}}$ and $\hat{\underline{a}} \leq \hat{\underline{b}}$. Since $\underline{a}^* \leq \hat{\underline{a}}$, the theorem holds.

Theorem 3.9. The set of all periodic classes of L is partially ordered under $\mathcal{F}$.

Proof. Let $\underline{B}(i)$, $\underline{B}(j)$ and $\underline{B}(k)$ be periodic classes of L such that $\underline{B}(i) \mathcal{F} \underline{B}(j)$ and $\underline{B}(j) \mathcal{F} \underline{B}(k)$. If $\underline{a}(i)$ is any element of $\underline{B}(i)$, by Lemma 3.2 there exists an element $\underline{a}(j)$ of $\underline{B}(j)$ such that $\underline{a}(i) \leq \underline{a}(j)$, and there exists an element $\underline{a}(k)$ of $\underline{B}(k)$ such that $\underline{a}(j) \leq \underline{a}(k)$. Hence for each member $\underline{a}(i)$ of $\underline{B}(i)$, there exists an element $\underline{a}(k)$ of $\underline{B}(k)$ such that $\underline{a}(i) \leq \underline{a}(j) \leq \underline{a}(k)$, so that $\underline{B}(i) \mathcal{F} \underline{B}(k)$, and $\mathcal{F}$ is transitive.

Let $\underline{B}(i) \mathcal{F} \underline{B}(j)$ and $\underline{B}(j) \mathcal{F} \underline{B}(i)$, where $\underline{B}(i)$ and $\underline{B}(j)$ are periodic classes of L . Lemma 3.2 implies that there exist elements $\underline{a}(1)$ and $\underline{a}(2)$ in $\underline{B}(i)$,

and $\underline{b}(1)$ and $\underline{b}(2)$ in $\underline{B}(j)$ such that $\underline{a}(1) \leq \underline{b}(1) \leq \underline{a}(2) \leq \underline{b}(2)$. From the definition of periodic class, $\underline{a}_h(1) = \underline{a}_h(2) = \underline{b}_h(1)$ whenever $h \notin M(\underline{B}(i))$, and $\underline{b}_h(1) = \underline{b}_h(2) = \underline{a}_h(2)$ whenever $h \notin M(\underline{B}(j))$. Thus $\underline{a}_h(2) = \underline{b}_h(1)$ whenever $h \notin M(\underline{B}(j)) \cap M(\underline{B}(j))$, which is the hypothesis of Theorem 3.1. The conclusion of the theorem, that $\hat{L}(\underline{a}(2)) = \hat{L}(\underline{b}(1))$, then implies that $M(\underline{B}(i)) = M(\underline{B}(j))$. From the definition of periodic class, $\underline{a}(2)$ and $\underline{b}(1)$ are members of the same periodic class and $\underline{B}(i) = \underline{B}(j)$. This demonstrates the trichotomy, and $\mathcal{F}$ is a partial ordering.

If L is assumed to be closed under $\vee$, then any pair of periodic classes of L has a least upper bound under $\mathcal{F}$. Such a property implies that if the number of periodic classes of L is finite, and if there exists a periodic class $\underline{B}(1)$ such that $\underline{B}(1) \mathcal{F} \underline{B}$ for every periodic class $\underline{B}$ of L , then the set of periodic classes of L forms a lattice. These statements are the content of Theorem 3.10 and Corollary 3.3.

Theorem 3.10. Let $\underline{B}(i)$ and $\underline{B}(j)$ be periodic classes of L , where L is closed under $\vee$. Then there is a least upper bound for $\underline{B}(i)$ and $\underline{B}(j)$ with respect to $\mathcal{F}$.

Proof. If $\underline{B}(i)$ and $\underline{B}(j)$ are periodic classes of L , let $\underline{B}(i) \vee \underline{B}(j)$ be the totality of elements of the form $\underline{a} \vee \underline{b}$, where $\underline{a} \in \underline{B}(i)$ and $\underline{b} \in \underline{B}(j)$.

If $\underline{a}$, $\underline{b}$ are any elements of $\underline{B}(i)$ and $\underline{c}$ and $\underline{d}$ are in $\underline{B}(j)$, then $(\underline{a} \vee \underline{c})_h = (\underline{b} \vee \underline{d})_h$ for all h which are in neither $M(\underline{B}(i))$ nor $M(\underline{B}(j))$. If $\underline{B}(h)$ and $\underline{B}(k)$ contain $\underline{a} \vee \underline{c}$ and $\underline{b} \vee \underline{d}$ respectively, $M(\underline{B}(i)) \cup M(\underline{B}(j)) \subseteq M(\underline{B}(h))$, $M(\underline{B}(k))$, from Theorem 2.6. The elements $(\underline{a} \vee \underline{c})$ and $(\underline{b} \vee \underline{d})$ satisfy the hypotheses of Theorem 3.1 with respect to $M(\underline{B}(h))$ and $M(\underline{B}(k))$, so that $\underline{B}(h) = \underline{B}(k)$. Thus $\underline{B}(i) \vee \underline{B}(j)$ belongs to some periodic class $\underline{B}(k)$.

From this definition, $\underline{B}(i) \mathcal{F} \underline{B}(k)$ and $\underline{B}(j) \mathcal{F} \underline{B}(k)$, so that $\underline{B}(k)$ is an upper bound for $\underline{B}(i)$ and $\underline{B}(j)$. If $\underline{B}(h)$ is also an upper bound for $\underline{B}(i)$ and $\underline{B}(j)$, then for any $\underline{a}$ in $\underline{B}(i)$ and any $\underline{b}$ in $\underline{B}(j)$, there is an element $\underline{c}$ in $\underline{B}(h)$ such that $\underline{c} \geq \underline{a}$ and $\underline{c} \geq \underline{b}$. Thus $\underline{c} \geq \underline{a} \vee \underline{b}$, and $\underline{B}(k) \mathcal{F} \underline{B}(h)$. This shows that $\underline{B}(k)$ is the least upper bound for $\underline{B}(i)$ and $\underline{B}(j)$.

Corollary 3.3. Let L have property II and be closed under $\vee$. If there is a periodic class $\underline{B}(1)$ of L such that $\underline{B}(1) \mathcal{F} \underline{B}$ for each periodic class $\underline{B}$ of L , then the set of periodic classes of L forms a lattice under $\mathcal{F}$.

Proof. If L has a finite number of periodic classes and is closed under $\vee$, Theorem 3.10 shows that any subset of L has a least upper bound under $\mathcal{F}$.

If $\underline{B}(i)$ and $\underline{B}(j)$ are any two periodic classes of L , then the set of periodic classes $\{\underline{B}\}$ of L for which $\underline{B} \mathcal{F} \underline{B}(i)$ and $\underline{B} \mathcal{F} \underline{B}(j)$ is non-void since $\underline{B}(1)$ is a member. Thus this set has a least upper bound, $\underline{B}(k)$, since $\{\underline{B}\}$ is a subset of the finite set of all periodic classes of L . If $\underline{B}(h) \mathcal{F} \underline{B}(i)$ and $\underline{B}(h) \mathcal{F} \underline{B}(j)$ then $\underline{B}(h) \mathcal{F} \underline{B}(k)$ from the least upper bound property of $\underline{B}(k)$ so that $\underline{B}(i)$ and $\underline{B}(j)$ have a greatest lower bound. Thus the set of periodic classes of L is a lattice with respect to $\mathcal{F}$.

Theorem 3.11 demonstrates that $\lambda(q(m, \underline{B}), \underline{B})$ is a non-decreasing function in the set of $\underline{B}^*$ such that $q(m, \underline{B}^*) = q(m, \underline{B})$ and $\underline{B} \mathcal{F} \underline{B}^*$.

Theorem 3.11. Let L be closed under $\vee$. Let $\underline{B}$ and $\underline{B}^*$ be periodic classes of L for which $\underline{B} \mathcal{F} \underline{B}^*$ and $q(m, \underline{B}) = q(m, \underline{B}^*)$. Then $\lambda(q(m, \underline{B}), \underline{B}) \leq \lambda(q(m, \underline{B}^*), \underline{B}^*)$.

Proof. Since $\underline{B} \not\cong \underline{B}^*$, there exist elements $\underline{a}, \underline{b}$ of $\underline{B}$ and $\underline{B}^*$ respectively, such that $\underline{a} \leq \underline{b}$. The construction of Theorem 3.1 may be employed to construct an element $\underline{a}^*$ such that $\underline{a} + kq(m, \underline{B}) = \underline{a}^*$ and $\underline{a}^*_h \geq \underline{b}_h$ for each h in $M(q(m, \underline{B}))$. The element $\underline{a}^* \vee \underline{b} = \underline{b}^*$ is in L , and is also in $\underline{B}^*$, from the definition of periodic class and the assumption that $q(m, \underline{B}) = q(m, \underline{B}^*)$. Then $\underline{a}^* \leq \underline{b}^*$ and $\underline{a}^*_h = \underline{b}^*_h$ for each h in $M(q(m, \underline{B}))$.

If $\underline{y} \in Y(\underline{a}^*, q(m, \underline{B}))$, then $\underline{y} + \underline{a}^* \in L$, and $\underline{y} + \underline{b}^* = (\underline{y} + \underline{a}^*) \vee \underline{b}^* \in L$.

This implies that $\underline{y} \in L(\underline{b}^*)$, and that $Y(\underline{a}^*, q(m, \underline{B})) \subseteq Y(\underline{b}^*, q(m, \underline{B}))$. The conclusion follows from the definition of $\lambda(q(m, \underline{B}), \underline{B})$.

In this chapter the concept of periodic class has been introduced as a partitioning of the elements of L . The principal result contained in Theorem 3.8, shows that whenever any components of two members of a periodic class differ by more than a fixed multiple of the periodicity vector, all components whose indices are spanned by this vector must differ by at least the corresponding component of the periodicity vector. Thus this concept will be important in the proof that any L which has properties I and II is the C-state lattice of a semi-modular circuit.

CHAPTER IV

THE CLASS OF C-STATE LATTICES

The principal result of this chapter is that any subset of $[I(-\infty, \infty)]^n$ which possesses properties I and II is the C-state lattice of a semi-modular circuit, and conversely. The proof is accomplished in several steps. After defining property I and showing that it implies semi-modularity, the "function theorem" is proven. The "function theorem" demonstrates a sufficient condition that a lattice with property I corresponds to the C-state lattice of a semi-modular circuit. The tools developed in Chapters II and III are employed to show that the sufficiency condition may be satisfied whenever L is a subset of $[I(0, \infty)]^n$ and has properties I and II. The converse is also established. A theorem which relates the concepts of semi-modular circuit theory to the concepts of Chapters II and III concludes this chapter.

I. PROPERTY I AND SEMI-MODULARITY

Theorem 1.12 states that all C-state lattices of semi-modular circuits are semi-modular. Thus any set of conditions imposed upon a set L of vectors must imply the semi-modularity of L in order to imply that L is the C-state lattice of a semi-modular circuit. The set of conditions listed in property I implies that a set L which satisfies these conditions is a semi-modular lattice.

Definition 4.1. The set L is defined to have property I if:

1. $L \subseteq [I(0, \infty)]^n$,
2. L is closed under $\vee$,
3. $\underline{0}$ is an element of L ,
4. Whenever $\underline{a}$ and $\underline{b}$ are elements of L such that $\underline{a}$ covers $\underline{b}$ in L , then $\underline{a}$ covers $\underline{b}$ in $[(0, \infty)]^n$.

Conditions 1 and 3 imply that $\underline{0}$ is the least element of L . An arbitrary translation $\underline{a} + \underline{x}$ of all elements $\underline{a}$ of L by some fixed element $\underline{x}$ of $[I(-\infty, \infty)]^n$ preserves the lattice and semi-modularity properties, from the definitions of the operations involved.

Theorem 4.1. If L has property I, then L is a semi-modular lattice.

Proof. If L has property I, then for any $\underline{x}$ and $\underline{y}$ in L , $\underline{x} \vee \underline{y}$ is in L .

From the definition of $\vee$, $\underline{x} \vee \underline{y} \geq \underline{x}$ and $\underline{x} \vee \underline{y} \geq \underline{y}$, so that $\underline{x} \vee \underline{y}$ is an upper bound for $\underline{x}$ and $\underline{y}$. If $\underline{z} < \underline{x} \vee \underline{y}$, then $\underline{z}_i < \max(\underline{x}_i, \underline{y}_i)$ for some i in $I(1, n)$, which means that $\underline{z}_i < \underline{x}_i$ or $\underline{z}_i < \underline{y}_i$. If $\underline{z}_i < \underline{x}_i$, then $\underline{z} \not\geq \underline{x}$, so that $\underline{z}$ is not an upper bound for $\underline{x}$ and $\underline{y}$. Since a similar argument holds if $\underline{z}_i < \underline{y}_i$, $\underline{x} \vee \underline{y}$ is the least upper bound for $\underline{x}$ and $\underline{y}$.

If $\underline{x} \in L$, let $A(\underline{x}) = \{\underline{b} \mid \underline{b} \in L, \text{ and } \underline{b} \leq \underline{x}\}$. If L has property I, $A(\underline{x})$ is finite for any $\underline{x}$ in L , since the number of elements $\underline{b}$ of $[I(0, \infty)]^n$ which satisfy $\underline{b} \leq \underline{x}$ is $\prod_{i \in I(1, n)} (\underline{x}_i + 1)$. Let $\underline{a}$ and $\underline{b}$ be arbitrary members of L . Then the set-theoretic intersection $A(\underline{a}) \cap A(\underline{b})$ is finite and is non-empty, since $\underline{0} \leq \underline{a}$ for any $\underline{a}$ in L . From property I, $\underline{d} = \bigvee_{\underline{c} \in (A(\underline{a}) \cap A(\underline{b}))} \underline{c}$ is in L , since the number of such $\underline{c}$'s is finite. For any $\underline{c}$ in $A(\underline{a}) \cap A(\underline{b})$, $\underline{c}_i \leq \underline{a}_i$ and $\underline{c}_i \leq \underline{b}_i$, so that $\underline{d}$ also satisfies $\underline{d}_i \leq \underline{a}_i$ and $\underline{d}_i \leq \underline{b}_i$. Thus $\underline{d}$ is a lower bound for $\underline{a}$ and $\underline{b}$. If $\underline{x}$ in L satisfies $\underline{x} > \underline{d}$, then $\underline{x} \not\leq \underline{a}$ or $\underline{x} \not\leq \underline{b}$, since $\underline{d}$ is the least upper bound of all elements which satisfy $\underline{x} \leq \underline{a}$ and $\underline{x} \leq \underline{b}$. Thus $\underline{x}$ is not a lower bound for $\underline{a}$ and $\underline{b}$, and $\underline{d}$ is the greatest lower bound. This shows that L is a lattice.

To demonstrate that L is a semi-modular lattice, it suffices to show that if $\underline{x}$, $\underline{y}$ and $\underline{z}$ are elements of L for which $\underline{x} \neq \underline{y}$, $\underline{x}$ covers $\underline{z}$, and $\underline{y}$ covers $\underline{z}$, then $\underline{x} \vee \underline{y}$ covers $\underline{x}$ and $\underline{y}$. From condition 4 of property I, there exist distinct indices i and j in $I(1, n)$ such that $\underline{x}_k = \underline{z}_k$ whenever $k \neq i$, $k \in I(1, n)$, $\underline{x}_i = \underline{z}_i + 1$, $\underline{y}_k = \underline{z}_k$, whenever $k \neq j$, and $\underline{y}_j = \underline{z}_j + 1$. This implies that

$\underline{x} \vee \underline{y}$ is defined by $(\underline{x} \vee \underline{y})_k = \underline{x}_k = \underline{y}_k = \underline{z}_k$ whenever $k \neq i$ or $k \neq j$, and $(\underline{x} \vee \underline{y})_i = \underline{x}_i = \underline{z}_i + 1 = \underline{y}_i + 1$ and $(\underline{x} \vee \underline{y})_j = \underline{y}_j = \underline{z}_j + 1 = \underline{x}_j + 1$. Thus $(\underline{x} \vee \underline{y})$ covers $\underline{x}$ and $\underline{y}$, and L is semi-modular.

The proof that L is a lattice does not use condition 4, but the following example shows that conditions 1, 2, and 3 alone do not imply semi-modularity. Let L consist of the elements 00, 10, 11, 02 and 12. The graph of the set is shown in Figure 4 below. Clearly, conditions 1, 2, and 3 of property I hold. The elements 10 and 02 both cover 00, but $10 < 11 < 12$ and 12 does not cover 10. Thus L is non-semi-modular.

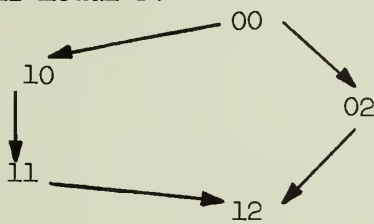


Figure 4

Example of a Non-semi-modular Lattice

Theorem 4.1 does not show that whenever L has property I, it is the C-state lattice of a semi-modular circuit. Theorem 4.2 displays a condition sufficient to imply this conclusion.

Theorem 4.2. Let L have property I and let the following three conditions hold:

1. For each i in $I(1, n)$ there is a function g_i which maps L onto $I(0, k_i - 1)$ such that if $\underline{x} \in L$, $\underline{y} \in L$ and $\underline{x}_i = \underline{y}_i$, then $g_i(\underline{x}) = g_i(\underline{y})$.
2. If $\underline{x}$ and $\underline{x} + \underline{e}(i)$ are elements of L , then $g_i(\underline{x}) \neq g_i(\underline{x} + \underline{e}(i))$.
3. If $\underline{x}$ and $\underline{y}$ are elements of L , $g_i(\underline{x}) = g_i(\underline{y})$ for each i in $I(1, n)$, and $\hat{\underline{x}} = \underline{x} + \underline{e}(j)$ is in L for some j in $I(1, n)$, then $\hat{\underline{y}} = \underline{y} + \underline{e}(j)$ is in L and $g_j(\hat{\underline{x}}) = g_j(\hat{\underline{y}})$.

Let T equal $\prod_{i \in I(1,n)} I(0, k_i - 1)$. Then there is a set $\{f_i\}$ of functions such that f_i maps T onto $I(0, k_i - 1)$ and an element u in T such that $\{f_i\}$ and u define a circuit C which is s.m. $[u]$, and whose C -state lattice is L .

Proof. Let L have property I and admit functions g_i , for each i in $I(1,n)$, which satisfy conditions 1, 2, and 3. Let S denote the subset $\{x\}$ of T for which there is at least one element $\underline{x}$ in L such that $g(\underline{x}) = x$, where $g(\underline{x})$ is the vector whose i th component is $g_i(\underline{x})$. For any element s of S , let $\underline{s}$ be any element of L for which $g_i(\underline{s}) = s_i$. If s is in S , let s' be defined by $s'_i = g_i(\underline{s} + \underline{e}(i))$ if $\underline{s} + \underline{e}(i)$ is in L , and $s'_i = s_i$ otherwise. Conditions 1 and 3 imply that the definition of s' is independent of $\underline{s}$, and may be written $s'_i = f_i(s)$, as s ranges over S' . Let u denote the vector defined by $u = g(\underline{0})$.

To prove the first assertion of the theorem it suffices to show that the circuit C defined by the set $\{f_i\}$ of functions is semi-modular with respect to u . Thus if u, a and b are elements of S for which $u \mathcal{F} a R b$, then it is to be shown that $b R a'$.

If $a R b$, then for any $\underline{a}$ in L for which $g(\underline{a}) = a$, let M be the set of indices j such that $b_j \neq a_j$. Then there exists an element $\underline{b} = \bigvee_{j \in M} (\underline{a} + \underline{e}(j))$ in L . If $\underline{a} + \underline{e}(j)$ is in L and $\underline{a}_j = \underline{b}_j$, then $(\underline{a} + \underline{e}(j)) \vee \underline{b}$ is in L , by closure under $\vee$, and $(\underline{a} + \underline{e}(j)) \vee \underline{b} = \underline{b} + \underline{e}(j)$. This implies that $a'_j = g_j(\underline{a} + \underline{e}(j)) = g_j(\underline{b} + \underline{e}(j)) = b'_j$. By closure under $\vee$, $\bigvee_j (\underline{b} + \underline{e}(j)) \in L$, where j ranges over the subset of $I(1,n)$ for which $a_j = b_j \neq a'_j$. Since $(\bigvee_j (\underline{b} + \underline{e}(j)))_k = \underline{b}_k$ if $b_k = a'_k$ and $(\bigvee_j (\underline{b} + \underline{e}(j)))_k = \underline{b}_k + 1$ if $b_k \neq a'_k$, the above statements imply that $g(\bigvee_j (\underline{b} + \underline{e}(j))) = a'$. From the representation of a' as $g(\bigvee_j (\underline{b} + \underline{e}(j)))$, $b R a'$ and the assertion holds.

Since C is semi-modular with respect to u , a C -state lattice L^* may be associated with C . The second assertion is that, considering L and L^* as subsets of $[I(0, \infty)]^n$, $L^* = L$. The theorem is proved by induction on the sum

of the components of the vectors. The vector $\underline{0}$ is in L by hypothesis and is the length of the partial allowed sequence consisting of u alone. Thus let the sets $\{\underline{a}\}, \{\underline{a}^*\}$ of L and of L^* such that $\sum_{i \in I(1,n)} \underline{a}_i \leq k$ and $\sum_{i \in I(1,n)} \underline{a}_i^* \leq k$ coincide. If $\underline{a}$ is an element of L such that $\sum_i \underline{a}_i = k + 1$, then by condition 4 of property I, there exists an element $\hat{\underline{a}}$ in L for which $\hat{\underline{a}} + \underline{e}(h) = \underline{a}$, and $\sum_i \hat{\underline{a}}_i = k$. This last statement implies that $\hat{\underline{a}}^*$ is an element of L^* .

Since $\underline{a}$ and $\hat{\underline{a}}$ are elements of L , there exist elements a and $\hat{a}$ of S such that $g(\underline{a}) = a$ and $g(\hat{\underline{a}}) = \hat{a}$. The statement that $\hat{\underline{a}} + \underline{e}(h) = \underline{a}$ implies that $\hat{a}_h = g_h(\hat{\underline{a}}) \neq g_h(\underline{a}) = a_h$ by condition 3 of the theorem, and that $\hat{a}_j = a_j$ for all j not equal to h . From the definition of the R relation, $\hat{a} R a$, and since $\hat{\underline{a}}^*$ is in L^* , $\underline{a}^* = L[u, \dots, \hat{a}] + L[\hat{a}, a] = \hat{\underline{a}}^* + \underline{e}(h) = \underline{a}$. Thus the set of elements $\underline{a}$ of L for which $\sum_i \underline{a}_i \leq k + 1$ is contained in the set of vectors $\underline{a}^*$ of L^* which satisfy $\sum_i \underline{a}_i^* \leq k + 1$.

To demonstrate the reverse inclusion, let $\underline{a}^*$ be an element of L^* which satisfies the relation $\sum_i \underline{a}_i^* = k + 1$. Furthermore, let the subsets of elements $\{\underline{b}^*\}, \{\underline{b}\}$ of L^* and of L for which $\sum_i \underline{b}_i^* \leq k$, and $\sum_i \underline{b}_i \leq k$, coincide. Since $\underline{a}^*$ is in L^* , there exists a sequence $\underline{a}^*(i)$ of elements of L^* , such that $t(\underline{a}^*(k)) R t(\underline{a}^*(k+1))$ for each k such that $\underline{a}^*(k)$ and $\underline{a}^*(k+1)$ are elements of the sequence. Let $\hat{\underline{a}}^*$ be the member of the sequence for which $t(\hat{\underline{a}}^*) R t(\underline{a}^*)$, and $t(\hat{\underline{a}}^*) \neq t(\underline{a}^*)$. This implies that $\underline{a}^* > \hat{\underline{a}}^*$, hence that $\sum_i \hat{\underline{a}}_i^* \leq k$, and that $\hat{\underline{a}}^*$ is in L . From the definition of L^* , $\underline{a}_i^* = \hat{\underline{a}}_i^*$ or $\underline{a}_i^* = \hat{\underline{a}}_i^* + 1$ for each i in $I(1,n)$.

The statement that $\hat{\underline{a}}^* R \underline{a}^*$ in S means that there exist two elements $\underline{b}$ and $\hat{\underline{b}}$ in L such that $g(\underline{b}) = a$ and $g(\hat{\underline{b}}) = \hat{a}$, and $\underline{b} = \bigvee_j (\hat{\underline{b}} + \underline{e}(j))$, where the union is taken over in those indices j for which $\underline{a}_j^* \neq \hat{\underline{a}}_j^*$. This implies that for each j for which $\underline{a}_j^* \neq \hat{\underline{a}}_j^*$, there exists an element $\hat{\underline{b}} + \underline{e}(j)$ in L , such that $g(\hat{\underline{b}}) R g(\hat{\underline{b}} + \underline{e}(j))$. Condition 3 of the hypothesis and the two statements that $\underline{a}$ is in L , and $g(\hat{\underline{a}}) = g(\hat{\underline{b}}) = \hat{a}$, imply that $\hat{\underline{a}} + \underline{e}(j)$ is a member of L ,

and that $g(\hat{\underline{a}} + \underline{e}(j)) = g(\hat{\underline{b}} + \underline{e}(j))$ for each j for which $a_j^* = \hat{a}_j^* \neq \hat{a}_j^*$. From the property of closure of L under $\vee$, $\bigvee_j (\hat{\underline{a}} + \underline{e}(j))$ is an element of L , where j ranges over those indices for which $a_j^* = \hat{a}_j^* \neq \hat{a}_j^*$. Since $a_j^* = \hat{a}_j^* \neq \hat{a}_j^*$ implies that $\underline{a}_j^* = \hat{\underline{a}}_j^* + 1$, the equation $\bigvee_j (\hat{\underline{a}} + \underline{e}(j)) = \underline{a}$ holds, and $\underline{a}$ is in L . This completes the proof of the isomorphism, and hence of the theorem.

This theorem is used as an existence theorem in later applications, since the proof of semi-modularity reduces to the verification that a given set of functions g_i satisfies the three stated conditions. Without such a "function theorem", the proof of semi-modularity would be required for each individual circuit.

The conclusion of Theorem 4.2 states, in part, that L , which has property I and admits the set of functions g_i , and L^* , the C-state lattice, are identical. Theorem 4.3 shows that the properties required of the g_i are satisfied by the functions $t_i(\underline{a})$, so that these properties are necessary as well as sufficient.

Theorem 4.3. Let C be a semi-modular circuit with respect to some initial state u , and let L be its C-state lattice. Then the component functions $t_i(\underline{a})$ satisfy the three conditions of Theorem 4.2.

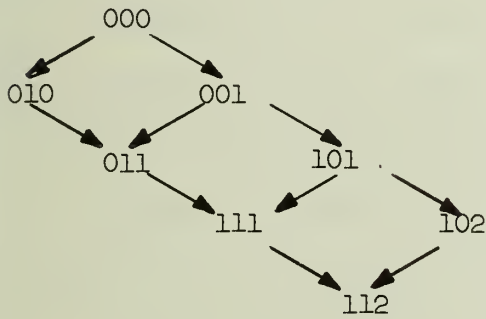
Proof. Let $\underline{a}$ and $\underline{b}$ be two elements of L , the C-state lattice of C , for which $\underline{a}_i = \underline{b}_i$. By Theorem 1.11, $\underline{a} \vee \underline{b}$ is in L if $\underline{a}$ and $\underline{b}$ are elements of L . Theorem 1.10 then implies that $\underline{a} \vee \underline{b} = L[u, \dots, a \dots t(\underline{a} \vee \underline{b})]$
 $= \underline{a} + L[a, \dots, t(\underline{a} \vee \underline{b})] = L[u, \dots, b, \dots, t(\underline{a} \vee \underline{b})] = \underline{b} + L[b, \dots, t(\underline{a} \vee \underline{b})]$. From the definition of L , $L_i[a, \dots, t(\underline{a} \vee \underline{b})] = L_i[b, \dots, t(\underline{a} \vee \underline{b})] = 0$, which implies that in these two partial allowed sequences $a_i = t_i(\underline{a} \vee \underline{b}) = b_i$. This is condition 1.

If $\underline{a}$ and $\underline{b}$ are elements of L such that $\underline{a} + \underline{e}(j) = \underline{b}$, for some j in $I(1,n)$, then there is a partial allowed sequence from $t(\underline{a})$ to $t(\underline{b})$ whose length $L[t(\underline{a}), \dots, t(\underline{b})]$ is equal to $\underline{e}(j)$. From Definition 1.21 of the length of a partial allowed sequence, and Theorem 1.13, $L[t(\underline{a}), \dots, t(\underline{b})] = \underline{e}(j)$ implies that $t_j(\underline{b}) = t_j(\underline{a}') \neq t_j(\underline{a})$.

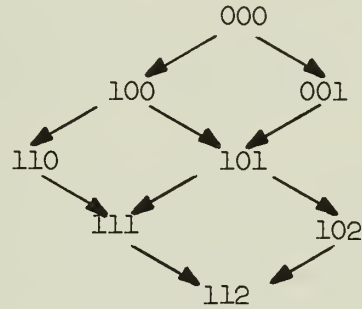
Condition 3 follows from the functional assumption that the set of states $\{b\}$ such that $a \mathcal{F} b$ is dependent only upon a .

Two simple examples demonstrate the direct application of Theorem 4.2. The two sets of elements of $[I(0,\infty)]^3$ shown in Figure 5 are the desired C-state lattices of two binary circuits, for which $u = 000$. Since the circuits are required to be binary, $g_i(\underline{a}) = g_i(\underline{a}_i)$ must satisfy

$$\begin{aligned} g_i(\underline{a}_i) &= 0 && \text{if } \underline{a}_i \text{ is even} \\ &= 1 && \text{if } \underline{a}_i \text{ is odd.} \end{aligned}$$



(a)



(b)

Figure 5

Example for Theorem 4.2

In the example of Figure 5a, the conditions of Theorem 4.2 are satisfied, since conditions 1 and 2 are immediate from the definition of $g_i(\underline{a})$. Condition 3 is vacuously satisfied, since no two elements of L have identical images under g .



In Figure 5b condition 3 is violated, since $g(100) = g(102)$ and $100 < 101$, but $102 \not\leq 103$, since $103 \notin L$. The set of functions

$$z'_1 = z_3 \vee z_1$$

$$z'_2 = 1$$

$$z'_3 = \bar{z}_1 \quad \text{generate the lattice of Figure 5a.}$$

II. THE SUFFICIENCY AND NECESSITY OF PROPERTIES I AND II

In this section the results of Chapters II and III are applied to demonstrate that a certain set $\{g_i\}$ of functions satisfies the conditions of Theorem 4.2 whenever L has properties I and II. Since the converse holds for any semi-modular circuit whose behavior is determined entirely from the set of functions and the initial state, these theorems characterize the set of C-state lattices of semi-modular circuits.

Theorem 4.4. Let L have properties I and II. Then there is a semi-modular circuit C whose C-state lattice L^* is isomorphic to L under the identity isomorphism.

Proof. It suffices to display a set of functions $g_i: L \rightarrow I[0, k_i - 1]$, for i in $I(1, n)$, which satisfy conditions 1, 2, and 3 of Theorem 4.2.

At the first step in the definition of such a set of functions, a vector $\underline{d}$ is formed. The vector $\underline{d}$ has the property that if $\underline{a}$ is an element of L and $\underline{a}_i \geq \underline{d}_i$, then node i is spanned in the periodic class which contains $\underline{a}$. For each periodic class $\underline{B}(k)$ of L , let $\underline{d}(k)$ be an element of $[I(0, \infty)]^n$ defined by

$$\begin{aligned} \underline{d}_i(k) &= 0 & \text{if } i \in M(\underline{B}(k)) \\ &= \underline{d}_i^* + 1 & \text{if } i \notin M(\underline{B}(k)), \end{aligned}$$

where $\underline{d}^*$ is any element of $\underline{B}(k)$. Since $\underline{d}_1^* = \underline{b}_1^*$ whenever $\underline{b}^*$ and $\underline{d}^*$ are any two members of $\underline{B}(k)$ and $i \notin M(\underline{B})$, $\underline{d}_i(k)$ is well defined. Next, let $\underline{d} = \bigvee_k \underline{d}(k)$, where k ranges over $I(1, r)$ and r is the number of periodic classes of L . If $\underline{a}_i \geq \underline{d}_i$, then $\underline{a}_i$ is greater than the maximum value which a non-spanned i th component may assume in any periodic class of L . Thus, the i th component index is spanned in the periodic class of L which contains $\underline{a}$.

The next step in the definition of the functions is to obtain a positive integer k_m^* for each periodicity vector $q(m, \underline{B}(r))$ of the final periodic class $\underline{B}(r)$. Let $\mu(m)$ be defined to be the least common multiple of $\mu(i, r; k, m)$ as i ranges over $I(1, r)$ and k over those elements of $M(\underline{B}(i))$ such that $\mu(i, r; k, m) \neq 0$.

Let $p(m)$ be the least member of $I(1, \infty)$ for which $p(m) \cdot \mu(m) > \lambda(q(k, \underline{B}(i), \underline{B}(i))) \cdot \mu(i, r; k, m)$, for each i in $I(1, r)$ and k in $M(\underline{B}(i))$. Then k_m^* is defined by

$$k_m^* = p(m) \cdot \mu(m).$$

For each i in $I(1, n)$, the integer k_i is defined by

$$k_i = \underline{d}_i + k_m^* q_i(m, \underline{B}(r)).$$

With these definitions, the functions g_i are defined by

$$\begin{aligned} g_i(\underline{a}) &= \underline{a}_i & \text{if } \underline{a}_i < \underline{d}_i \\ g_i(\underline{a}) &= \underline{d}_i + \text{Res } [\underline{a}_i - \underline{d}_i] \text{ mod } k_m^* q_i(m, \underline{B}(r)) & \text{if } \underline{a}_i \geq \underline{d}_i. \end{aligned}$$

To complete the proof, the three conditions of Theorem 4.2 must be verified. Conditions 1 and 2 are obviously satisfied from the component-wise definition of $g_i(\underline{a})$ and the observation that $k_m^* \geq 2$. The remaining problem is to show that if $g(\underline{a}) = g(\underline{b})$ and $\underline{a} + \underline{e}(j)$ is in L , then $\underline{b} + \underline{e}(j)$ is in L and $g_j(\underline{a} + \underline{e}(j)) = g_j(\underline{b} + \underline{e}(j))$. It suffices to show that $\underline{a}$ and $\underline{b}$ are members of some $\underline{B}(i)$ and that $\underline{b} = \underline{a} + \sum_{m \in I(1, n(\underline{B}(i)))} c_m q(m, \underline{B}(i))$. The sufficiency of statement is implied by Theorem 3.2, which states that $L(\underline{a}) = L(\underline{b})$, and by the component-wise definition of g_j .

Let $g_i(\underline{a}) = g_i(\underline{b}) = a_i$ for all i in $I(1, n)$. This implies that

$\underline{a}_i = \underline{b}_i$ if either $\underline{a}_i < \underline{d}_i$ or $\underline{b}_i < \underline{d}_i$, from the definition of $\underline{d}$. Furthermore, the set of indices i for which $\underline{a}_i = \underline{b}_i < \underline{d}_i$ contains the set of all indices unspanned by either $M(\underline{A}(\underline{a}))$ or $M(\underline{A}(\underline{b}))$. This statement implies that $\underline{a}$ and $\underline{b}$ satisfy the hypotheses of Theorem 3.1, and hence that $\widehat{L}(\underline{a}) = \widehat{L}(\underline{b})$. Thus the sets of unspanned indices of $\underline{A}(\underline{a})$ and of $\underline{A}(\underline{b})$ coincide, and $\underline{a}_i = \underline{b}_i$ whenever index i is unspanned in this common set. Thus the elements $\underline{a}$ and $\underline{b}$ are members of the same periodic class $\underline{B}(i)$.

By Theorem 3.6, $\underline{b}$ may be expressed as

$$\underline{b} = \underline{a} + \sum_j c_j q(j, \underline{B}(i)) + \sum_j y(j),$$

where j ranges over $I(1, n(\underline{B}(i)))$, $c_j \in I(-\infty, \infty)$, and $y(j) \in Y(\underline{a}, q(j, \underline{B}(i)))$.

From the definition of $Y(\underline{a}, q(j, \underline{B}(i)))$,

$$\underline{a} + \sum_j c_j q(j, \underline{B}(i)) \leq \underline{b} < \underline{a} + \sum_j [c_j + \lambda(q(j, \underline{B}(i)), \underline{B}(i))] \cdot q(j, \underline{B}(i))$$

$$\text{or} \quad 0 \leq \underline{b} - \underline{a} - \sum_j c_j q(j, \underline{B}(i)) < \lambda(q(j, \underline{B}(i)), \underline{B}(i)) \cdot q(j, \underline{B}(i)) \quad .$$

From the component-wise definition of all elements, the last equation represents the equation

$$0 = (\underline{b} - \underline{a} - \sum_j c_j q(j, \underline{B}(i)))_h \leq (\sum_j \lambda(q(j, \underline{B}(i)), \underline{B}(i)) \cdot q(j, \underline{B}(i)))_h$$

for each h in $I(1, n)$, with strict inequality on the right for at least one h .

Since for each h , there is at most one j for which $q_h(j, \underline{B}(i)) \neq 0$, the summations may be removed, yielding the inequality

$$0 \leq b_h - a_h - c_j(h) q_h(j(h), \underline{B}(i)) \leq \lambda(q(j(h), \underline{B}(i)), \underline{B}(i)) \cdot q_h(j(h), \underline{B}(i)) \quad .$$

The definition of g_i implies that, for each h for which $\underline{a}_i \geq \underline{d}_i$,

$$\lambda(q(j(h), \underline{B}(i)), \underline{B}(i)) \cdot q_h(j(h), \underline{B}(i)) \geq b_h - a_h - c_j(h) q_h(j(h), \underline{B}(i))$$

$$= t(h) k_m^* q_h(m, \underline{B}(r))$$

$$\begin{aligned}
&> t(h) \lambda(q(j(h), \underline{B}(i)), \underline{B}(i)) \mu(i, r; j(h), m) q_h(m, \underline{B}(r)) \\
&= t(h) \lambda(q(j(h), \underline{B}(i)), \underline{B}(i)) q_h(j(h), \underline{B}(i)).
\end{aligned}$$

This implies that $t(h) = 0$ for each h , and hence that

$$\underline{b} = \underline{a} + \sum_j c_j q(j, \underline{B}(i)).$$

This is the sufficient condition mentioned above, so that the theorem holds.

In the proof of this theorem, it has been shown that whenever $g(\underline{a}) = g(\underline{b})$, $\underline{b} = \underline{a} + \sum_j c_j q(j, \underline{B}(i))$, where $\underline{a}$ and $\underline{b}$ are elements of the periodic class $\underline{B}(i)$. The converse of the theorem is false, however, as the following example shows.

Let L be the lattice displayed in Figure 6, page 59. This lattice has the four periodic classes $\underline{B}(1) = \{0000\}$, $\underline{B}(2) = \{10xx\}$, $\underline{B}(3) = \{01xx\}$, and $\underline{B}(4) = \{11xx\}$, where the x entries represent spanned components. The periodicity vector of $\underline{B}(2)$ is 0032, and of $\underline{B}(3)$, 0023. The class $\underline{B}(4)$ has the two periodicity vectors 0010 and 0001. In the construction of theorem 4.4, $\underline{d} = 2211$, $k_3^* = 6$ and $k_4^* = 6$, so that $k_1 = 2$, $k_2 = 2$, $k_3 = 7$, and $k_4 = 7$. In this assignment, the elements 1 0 1 1, 1 0 19 13, 1 0 37 25, ... map onto 1 0 1 1; 1 0 7 5, 1 0 25 17, 1 0 43 29, ... onto 1 0 7 5; and 0 1 2 2, 0 1 14 20, 0 1 26 38, onto 0 1 2 2. In the first two sequences, the third component assumes the values 1 or 7 three times for every two times that node 4 assumes the values 1 or 5. The reverse statement is true for the value 2 in the third sequence.

The example shows that a common multiple of all periodicity vectors which span some node need not be found. In fact, it is in general impossible to find such a vector, since, as in this case, a multiple of 0032 is not a multiple of 0023.

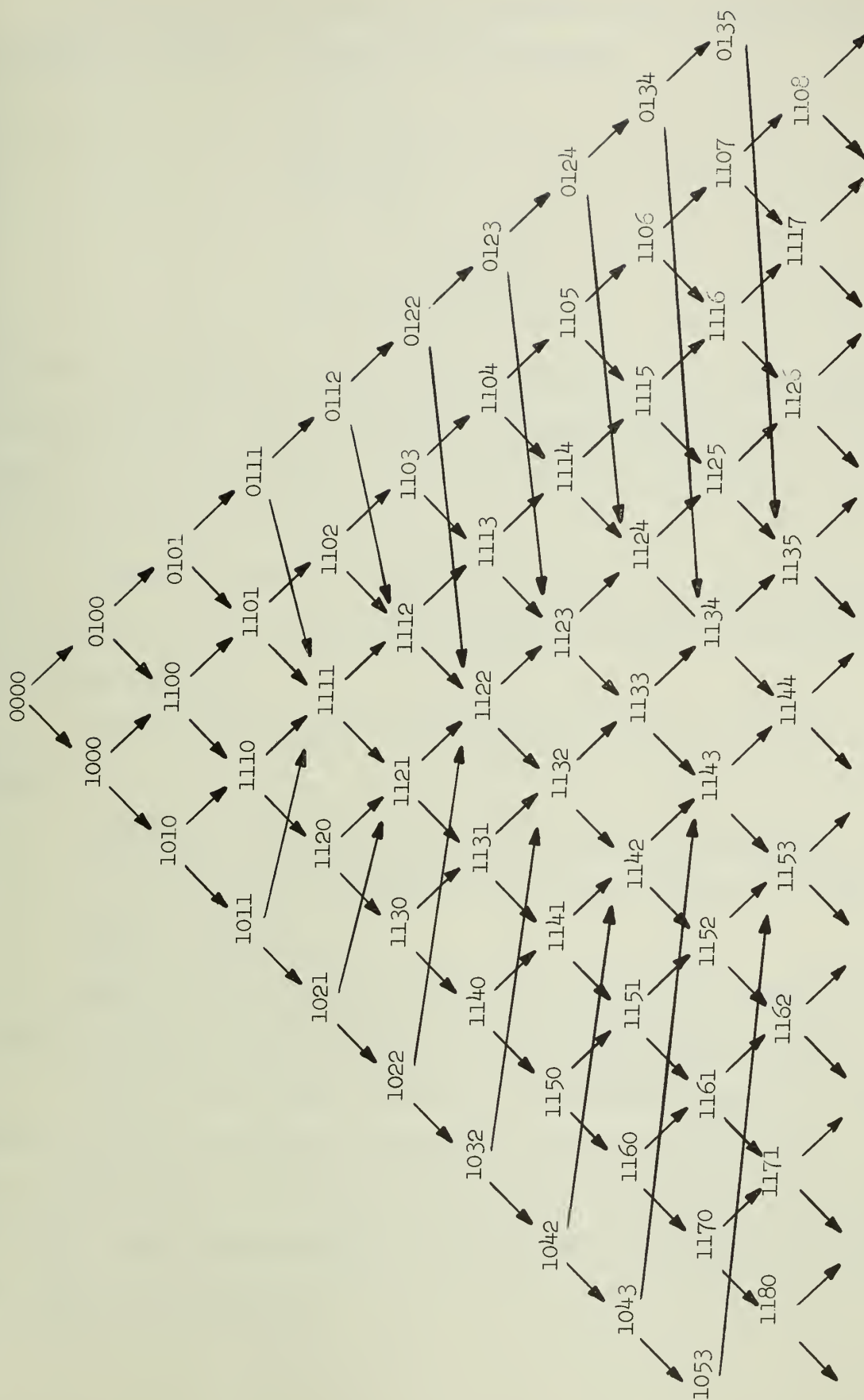


Figure 6

Counterexample to the Converse of Theorem 4.4

This construction does not necessarily yield a minimum number of states or images of L under g . In this case the functions

$$g_1(\underline{a}) = \underline{a}_1$$

$$g_2(\underline{a}) = \underline{a}_2$$

$$g_3(\underline{a}) = \text{Res } \underline{a}_3 \text{ mod } 6$$

$$g_4(\underline{a}) = \text{Res } \underline{a}_4 \text{ mod } 6$$

also yield a semi-modular circuit, from Theorem 4.2. In this case there are 144 possible states, while the original assignment yielded 198. The numbers of elements of S which are images of L are 97 under the new assignment, and 114 under the assignment of Theorem 4.4.

Another corollary to the proof of Theorem 4.4 is that for the set $\{g_i\}$, each element in the sequence $g_1(\underline{a}(0)), g_1(\underline{a}(1)) \dots g_1(\underline{a}(r))$, where $\underline{a}_1(j) = j$, has a unique successor.

The concluding theorem of this section contains the converse to Theorem 4.4.

Theorem 4.5. The class of all sets L which have properties I and II is identical with the class of all C -state lattices L^* of semi-modular circuits C .

Proof. Theorem 4.4 shows that any set L which has properties I and II is identical with the C -state lattice L^* of some semi-modular circuit C .

Let L^* be the C -state lattice of a semi-modular circuit C . Property I requires that L^* be contained in $[I(0, \infty)]^n$, that L^* be closed under $\vee$, that 0 be an element of L^* , and that $\underline{a}$ covers $\underline{b}$ in L^* implies that $\underline{a}$ covers $\underline{b}$ in $[I(0, \infty)]^n$. These four properties are the content of Theorems 1.11 through 1.13. Thus L^* has property I.

Each state of C corresponds to exactly one $\mathcal{S}$ -class of L^* by Theorem 1.14. Since the number of states of C is finite, then the number of $\mathcal{S}$ -classes of L^* is finite, and L^* has property II.

This same topic is discussed in Chapter 7 of [7], in which a theorem similar to 4.5 was proved.

III. THE RELATION BETWEEN PERIODIC AND CYCLIC CLASSES

The final topic of the chapter relates the periodic class as defined in Chapter III and employed in this chapter, to the cyclic class of a semi-modular circuit. In the preceding section, the property that $g(\underline{a}) = g(\underline{b})$ was shown to imply that $L(\underline{a}) = L(\underline{b})$. The theorem of this section shows that each cyclic class is contained in a periodic class, and that the cycling vectors, if any, are multiples of the periodicity vectors.

Theorem 4.6. Let $\underline{A}(i)$ be a cyclic class of L , the C -state lattice of a semi-modular circuit C . Then there is a periodic class $\underline{B}(i)$ of L which contains $\underline{A}(i)$. If $w(j, \underline{A}(i))$ is a cycling vector of $\underline{A}(i)$, then $w(j, \underline{A}(i)) = \eta(j, i) q(j, \underline{B}(i))$, where $\eta(j, i) \in I(1, \infty)$ and $q(j, \underline{B}(i)) \in Q(\underline{B}(i))$.

Proof. If $\underline{A}(i)$ has exactly one element $\underline{a}$, then $\underline{A}(i)$ is contained in a periodic class, as are all elements of L . Since the set of cycling vectors $w(j, \underline{A}(i))$ is vacuous if $\underline{A}(i)$ has exactly one element, the second conclusion holds.

An index j is defined to be spanned in $\underline{A}(i)$ if there is a partial allowed sequence from any element $\underline{a}$ of L to an element $\underline{b}$ of L such that $\underline{b}_j \neq \underline{a}_j$ and $t(\underline{a}) = t(\underline{b})$. Since Theorem 4.4 shows that $\underline{a} \mathcal{S} \underline{b}$, and $\underline{a}_j \neq \underline{b}_j$, there is an element $c_k q(k, \underline{B}(i))$ such that $\underline{b}_j - \underline{a}_j = c_k q_j(k, \underline{B}(i))$, so that node j is spanned in the

$\mathcal{L}$ -class of $\underline{a}$ and $\underline{b}$. From the definition of periodic class, node j is spanned in $Q(B(i))$. Since, by definition, for any $\underline{a}$ and $\underline{c}$ in $\underline{A}(i)$, $\underline{a}_h = \underline{c}_h$ for any h which is not spanned by $w(\underline{a})$, Theorem 3.1 and Definition 3.1 imply that $\underline{a}$ and $\underline{c}$ are members of the same periodic class, $\underline{B}(i)$.

If $t(\underline{a}) = t(\underline{b})$ and $\underline{b} - \underline{a} = w(h, \underline{A}(i)) = \sum_k c_k q(k, \underline{B}(i))$, it suffices to show that exactly one c_k is non-zero. Since property II and closure under $\vee$ are implied by Theorem 4.5, Corollary 2.5 implies that the $q(k, \underline{B}(i))$ have disjoint indices for their non-zero components. If $c_h q(h, \underline{B}(i)) \neq 0$, then $t(\underline{a}) = t(\underline{a} + c_h q(h, \underline{B}(i)))$, from Theorem 4.5, and the assumption that $w(h, \underline{A}(i))$ is a cycling vector implies that $w(h, \underline{A}(i)) = c_h q(h, \underline{B}(i))$. Thus $c_h = \mathcal{V}(h, i)$ is the desired parameter.

The converse of Theorem 4.6 does not hold, since two assignments of functions g_i to a lattice L with properties I and II may not assign elements of the same periodic class to the same cyclic class. Consider the lattice L which is diagrammed in Figure 7 below. One possible assignment is that

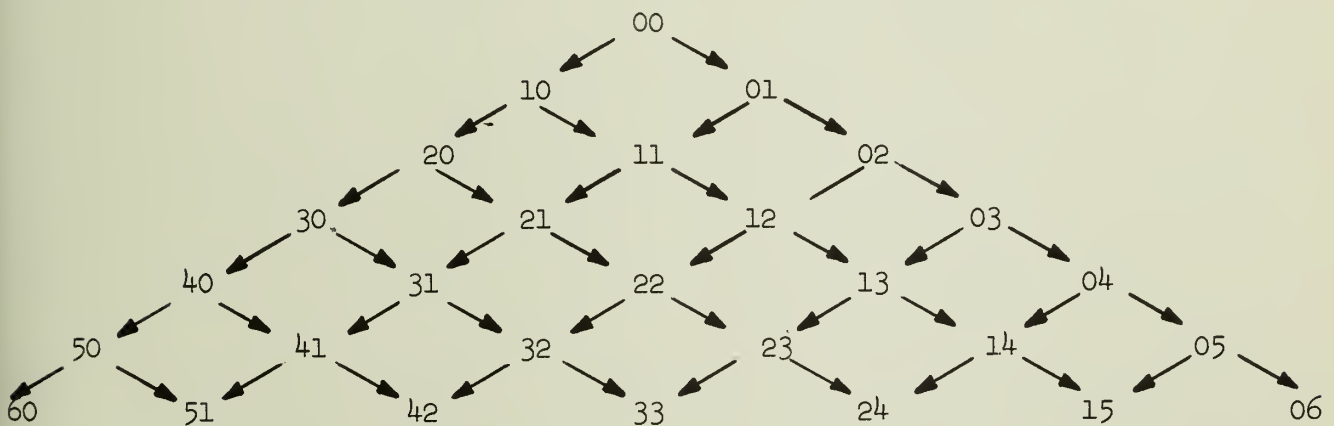


Figure 7

Counterexample to the Converse of Theorem 4.6

$g_i(\underline{a}) = \underline{a}_i \bmod 2$, for $i = 1$ and 2 . This leads to a circuit $z_1' = \bar{z}_1$, $z_2' = \bar{z}_2$ with initial state 00 . There is one cyclic class which is also a periodic class, with the two cyclic vectors (20) and (02) . Another assignment is $g_i(\underline{a}) = 0$ if $\underline{a}_i = 0$; 1 if $\underline{a}_i = 1 \bmod 2$; and 2 if $\underline{a}_i = 0 \bmod 2$ and $\underline{a}_i > 0$, for $i = 1$ and 2 . The cyclic classes consist of $\{(00)\}$, $\{(k0)\}$, $\{(0h)\}$ and $\{(k,h)\}$, as h and k range independently over $I(1, \infty)$. Thus the cyclic classes are not invariant under the assignment of the functions g_i .

In Chapter IV, properties I and II were shown to be necessary and sufficient conditions that L be the C-state lattice of some semi-modular circuit. The theorem was proved using the tools of Chapters II and III, the function theorem 4.2, and a particular set of function $\{g_i\}$. The rest of the chapter was devoted to deriving the relationships between C-state lattices of semi-modular circuits and the theoretical results of earlier chapters. In Chapter V, these results will be applied to yield a simpler way of representing the C-state lattice.

CHAPTER V

APPLICATIONS OF THE THEORY

In this chapter, the results of Chapter IV are applied to important problems in the design of switching circuits. The first such problem is that of substituting a simpler requirement on the problem specification than that of property I. The second problem is that of removing an approximation used in the transition from physical circuits to the mathematical model.

In the next section, the representation $\{S, B(S)\}$ of a lattice L with property I is defined. Such a representation is shown to be equivalent to property I, so that Theorem 4.4 may be rephrased to require that $\{S, B(S)\}$ be a representation and the corresponding L have property II. The relations between the elements of the representation and the concepts of join-irreducible elements and distributivity are also discussed.

In the second section, the problem of delays in the lines connecting the elements is considered. This problem is ignored in the transition from physical circuit to the general mathematical model of the circuit, but must be considered in the operation of a synthesized circuit..

I. THE B-REPRESENTATION

In Definition 4.1, condition 2 of property I states that L is closed under $\vee$. This implies that the decision procedure must form all possible combinations under $\vee$ of elements of L and examine them for membership in L . In practical problems, the synthesis process is required to satisfy conditions on the relative order of signal changes, and not, as a rule, a lattice. These two observations suggest that the specification of the problem be rephrased to require less than property I. Of course, the specifications must ensure that a lattice having property I may be formed.

In Definition 5.1, such a revised specification is prescribed.

Definition 5.1. A (B-)representation is defined to be an ordered pair $\{S, B(S)\}$, where S is a set and $B(S)$ is a function defined on elements of S . The elements of S are ordered pairs (α, i) , where $i \in I(1, n)$ for some fixed n in $I(1, \infty)$, $\alpha \in I(1, \infty)$, and $(\alpha-1, i) \in S$ if $\alpha > 1$. For each (α, i) in S , $B(\alpha, i)$ is a non-void subset of $[I(0, \infty)]^n$, which has the following properties:

1. If $\underline{a} \in B(\alpha, i)$, then $\underline{a}_i = \alpha$
2. If $\underline{a}$ and $\underline{b}$ are elements of $B(\alpha, i)$, then $\underline{a} \not\leq \underline{b}$ and $\underline{b} \not\leq \underline{a}$.
3. If $\underline{a} \in B(\alpha, i)$, then
 - a) for each j for which $j \neq i$ and $\underline{a}_j > 0$, there is an element $\underline{b}$ in $B(\underline{a}_j, j)$ such that $\underline{b} < \underline{a}$, and
 - b) if $\alpha > 1$, there is an element $\underline{c}$ in $B(\alpha-1, i)$ such that $\underline{c} < \underline{a}$.

Two representations $\{S_1, B_1(S_1)\}$ and $\{S_2, B_2(S_2)\}$ are equal if

$S_1 = S_2 = S$, and $B_1(\alpha, i) = B_2(\alpha, i)$ for each (α, i) in S .

Before proving the pertinent theorems, the definition may be partially motivated by demonstrating the relation between such representations and the set of join-irreducible elements of L .

Definition 5.2. For any set L , the set $A(\alpha, i)$ is defined by

$A(\alpha, i) = \{\underline{b} \mid \underline{b} \in L \text{ and } \underline{b}_i = \alpha\}$. If $A(\alpha, i) \neq \emptyset$, then the set $B(\alpha, i)$ is defined by $B(\alpha, i) = \{\underline{c} \mid \underline{c} \text{ is a minimal element of } A(\alpha, i)\}$.

Conditions 1, 2, and 3 of Definition 5.1 imply that elements of $B(\alpha, i)$

have the properties of elements $\underline{a}$ minimal with respect to the property that

$\underline{a}_i = \alpha$. Theorem 5.1 demonstrates the equivalence of the two concepts.

Theorem 5.1. Let L be closed under $\vee$. Then $\underline{a}$ in L is join-irreducible if and only if $\underline{a} \in B(\underline{a}_i, i)$ for at least one i in $I(1, n)$.

Proof. If $\underline{a} \in B(\underline{a}_i, i)$, the definition of minimal element implies that there is no element $\underline{c}$ in L for which $\underline{c} < \underline{a}$ and $\underline{c}_i = \alpha$. Thus if $\underline{c}, \underline{d}$, are in L , and $\underline{c} < \underline{a}, \underline{d} < \underline{a}$, then $(\underline{c} \vee \underline{d})_i < \underline{a}_i$, so that $(\underline{c} \vee \underline{d}) \neq \underline{a}$. This implies that $\underline{a}$ is join-irreducible.

If $\underline{a}$ is join-irreducible, then $\underline{a} \neq (\underline{c} \vee \underline{d})$, where $\underline{c} \in L, \underline{c} < \underline{a}, \underline{d} \in L$ and $\underline{d} < \underline{a}$. If for each index i , there is an element $\underline{b}(i)$ in L such that $\underline{b}(i) < \underline{a}$ and $\underline{b}_i(i) = \underline{a}_i$, then $\bigvee_{i \in I(1, n)} \underline{b}(i) = \underline{a}$, which contradicts the assumption that $\underline{a}$ is join-irreducible. Thus there is at least one index i in $I(1, n)$ such that for any $\underline{c}$ in L satisfying $\underline{c} < \underline{a}, \underline{c}_i < \underline{a}_i$. This implies that $\underline{a}$ is a minimal element of $A(\underline{a}_i, i)$ or $\underline{a} \in B(\underline{a}_i, i)$.

Lemma 5.1. Let $\{S, B(S)\}$ be a representation. If $\underline{a}$ and $\underline{b}$ are elements of $B(\alpha, i)$ and $B(\underline{a}_j, j)$ respectively, $i \neq j$, and $\underline{b} < \underline{a}$, then $\underline{a}_i > \underline{b}_i$.

If $\underline{a}$ is an element of $B(\alpha, i)$ and of $B(\beta, j)$ then $i = j$ and $\alpha = \beta$.

Proof. By property 3a of Definition 5.1, the hypothesis that

$\underline{b} \in B(\underline{a}_j, j)$ implies that there is an element $\underline{c}$ of $B(\underline{b}_i, i)$ such that $\underline{c} < \underline{b} < \underline{a}$. Thus $\underline{c}_i = \underline{b}_i \leq \underline{a}_i$. If $\alpha > \underline{a}_i = \underline{b}_i$, then property 1 of Definition 5.1 is violated, so that $\underline{a}_i > \underline{b}_i$.

In the second assertion, if $i = j$, then by property 1 of Definition 5.1, $\underline{a}_i = \alpha = \beta$. If $i \neq j$, then there is an element $\underline{b}$ in $B(\underline{a}_j, j) = B(\beta, j)$ such that $\underline{b} < \underline{a}$. Since $\underline{a}_j = \underline{b}_j$, $\underline{b}$ is an element of $B(\beta, j)$ which satisfies $\underline{b} < \underline{a}$, and $\underline{a}$ is not in $B(\beta, j)$, by condition 2 of Definition 5.1. This contradicts the hypothesis, the first case holds, and the assertion has been verified.

The principal result of the section is L has property I if and only if it has a representation in the sense of Definition 5.1. The correspondences between representations and sets with property I are defined in Definition 5.3. Lemmas 5.2 and 5.3 yield properties of the sets L derived by Definition 5.3a. Theorem 5.2 displays the principal result that the two mappings of Definition 5.3a and 5.3b yield sets with property I and representations respectively. This theorem also shows that the mappings are 1 -1 onto, between the set of all sets L which have property I and the set of all representations.

Definition 5.3.

- a) Let $\{S, B(S)\}$ be a representation. Then $L = L(\{S, B(S)\})$ is defined by $L = \{\underline{a} \mid \underline{a} \in [I(0, \infty)]^n, \text{ and for each } i \text{ in } I(1, n) \text{ for which } \underline{a}_i > 0, (\underline{a}_i, i) \in S \text{ and there is some element } \underline{b}(i) \text{ in } B(\underline{a}_i, i) \text{ such that } \underline{a} \geq \underline{b}(i)\}$.
- b) If L satisfies property I, then $\{S, B(S)\}(L)$ is defined by:
- (a) $(\alpha, i) \in S$ if and only if for some $\underline{a}$ in L ,
- $$\underline{a}_i = \alpha > 0, \text{ and}$$
- (b) if $(\alpha, i) \in S$, then $B(\alpha, i)$ is the set of minimal elements of the subset $\{\underline{b}\}$ of elements of L such that $\underline{b}_i \geq \alpha$.

Lemma 5.2. If $\{S, B(S)\}$ is a representation, and if $\underline{b} \in B(\alpha, i)$ for some (α, i) in S , then $\underline{b} \in L(\{S, B(S)\})$.

Proof. Since $\underline{b} \in B(\alpha, i)$, $\underline{b} \geq \underline{b}$ for some $\underline{b}$ in $B(\underline{b}_i, i)$. From property 3a of Definition 5.1, for each j different from i for which $\underline{b}_j > 0$, there is an element $\underline{b}(j)$ in $B(\underline{b}_j, j)$ such that $\underline{b}(j) < \underline{b}$. Since $B(\underline{d}_k, k)$ is defined only if $(\underline{d}_k, k)$ is in S , the assertion that $\underline{b}$ is in L follows from Definition 5.3a.

Lemma 5.3. If $\{S, B(S)\}$ is a representation and $\underline{a}$ and $\underline{b}$ are elements of $L = L(\{S, B(S)\})$, then $\underline{a} \vee \underline{b} \in L$.

Proof. If $\underline{a} \in L$, then Definition 5.1 implies that for each i in

$I(1, n)$ such that $\underline{a}_i > 0$, $(\underline{a}_i, i)$ is an element of S and there exists an $\underline{a}(i)$ in $B(\underline{a}_i, i)$ such that $\underline{a} \geq \underline{a}(i)$. The corresponding statement holds for $\underline{b}$ and $\underline{b}(i)$.

From the definition of $\underline{a} \vee \underline{b}$, $((\underline{a} \vee \underline{b})_i, i) \in S$, $\underline{a} \vee \underline{b} \geq \underline{a} \geq \underline{a}(i)$, and $\underline{a} \vee \underline{b} \geq \underline{b} \geq \underline{b}(i)$, so that $\underline{a} \vee \underline{b} \in L$, by Definition 5.3a.

Theorem 5.2. If $\{S, B(S)\}$ is a representation, then $L(\{S, B(S)\})$ has property I. If $\{S_1, B_1(S_1)\}$ and $\{S_2, B_2(S_2)\}$ are distinct, then $L(\{S_1, B_1(S_1)\}) \neq L(\{S_2, B_2(S_2)\})$.

If L has property I, then $\{S, B(S)\}(L)$ is a representation. If L^* has property I, then $L(\{S, B(S)\}(L^*)) = L^*$. If L_1 and L_2 are distinct, then $\{S, B(S)\}(L_1) \neq \{S, B(S)\}(L_2)$. If $\{S_1, B_1(S_1)\}$ is a representation then $\{S, B(S)\}(L(\{S_1, B_1(S_1)\})) = \{S_1, B_1(S_1)\}$.

Proof. To demonstrate the first assertion, the conditions of Definition 4.1 must be verified. It is obvious that $L \subseteq [I(0, \infty)]^n$, since each $\underline{a}$ in L satisfies $\underline{a} \in [I(0, \infty)]^n$. The element $\underline{0}$ is in L , since the conditions of Definition 5.3a are satisfied vacuously. The set L is closed under $\vee$, by Lemma 5.3.

The final step in the verification is to show that covering in L implies covering in $[I(0, \infty)]^n$. Let $\underline{a}$ and $\underline{b}$ be elements of L for which $\underline{b}$ covers $\underline{a}$, and let M be the set of indices j such that $\underline{a}_j < \underline{b}_j$. For each k in M , the hypothesis that $\underline{b} \in L$ implies that there is an element $\underline{d}(k)$ in $B(\underline{b}_k, k)$ such that $\underline{b} \geq \underline{d}(k)$. Since $\underline{d}(k) \in L$ by Lemma 5.2 and L is closed under $\vee$ by Lemma 5.3, then $\underline{a} \vee \underline{d}(k) \in L$ and $\underline{a} \leq \underline{a} \vee \underline{d}(k) \leq \underline{b}$. Since $\underline{d}_k(k) = \underline{b}_k > \underline{a}_k$, the covering hypothesis implies that $\underline{a} \vee \underline{d}(k) = \underline{b}$. Thus, $\underline{d}_j(k) = \underline{d}_j(h) = \underline{b}_j$ for all j, k and h in M .

If k_1 and k_2 are distinct elements of M , then by Lemma 5.1, there is an element $\underline{f}$ in $B(\underline{d}_{k_2}(k_1), k_2)$ such that $\underline{f} < \underline{d}(k_1)$ and $\underline{f}_{k_1} < \underline{d}_{k_1}(k_1) = \underline{b}_{k_1}$.

Since $\underline{f}_{k_2} = \underline{d}_{k_2}(k_1) = \underline{b}_{k_2} > \underline{a}_{k_2}$ by property 1 of Definition 5.1,

$\underline{a} < \underline{a} \vee \underline{f} < \underline{d}(k_1) \vee \underline{a} = \underline{b}$. This contradicts the covering hypothesis, and M has exactly one element.

Let k be that index for which $\underline{b}_k > \underline{a}_k$. Since $\underline{b} \in L$, $(\underline{b}_k, k) \in S$. If $\underline{b}_k > 1$, then $(\underline{b}_k - 1, k) \in S$, and there exists an element $\underline{c}$ in $B(\underline{b}_k - 1, k)$ for which $\underline{c} < \underline{b}$. Thus $\underline{a} \leq \underline{a} \vee \underline{c} < \underline{b}$, which implies that $\underline{a} \vee \underline{c} = \underline{a}$, by the covering hypothesis. Since $\underline{b}_k > (\underline{a} \vee \underline{c})_k = \underline{a}_k = \underline{b}_k - 1$, this is also a covering in $[I(0, \infty)]^n$. If $\underline{b}_k = 1$, then $\underline{a}_k = 0$, and the same conclusion holds. Thus L has property I.

To show that the mapping of $\{S, B(S)\}$ onto $L(\{S, B(S)\})$ is 1 - 1, let $\{S_1, B_1(S_1)\}$ and $\{S_2, B_2(S_2)\}$ be distinct representations. If $(\alpha, i) \in S_1$ but $(\alpha, i) \notin S_2$, then any element $\underline{b}$ of $B_1(\alpha, i)$ is in $L_1 = L(\{S_1, B_1(S_1)\})$. The vector $\underline{b}$ is not in $L_2 = L(\{S_2, B_2(S_2)\})$, since no element $\underline{b}$ of L_2 may satisfy $\underline{b}_i \geq \alpha$. If $S_1 = S_2$, $\underline{b} \in B_1(\alpha, i)$, but $\underline{b} \notin B_2(\alpha, i)$, then the assertion holds if $\underline{b} \notin L_2$. If $\underline{b} \in L_2$, then the hypothesis that $\underline{b} \notin B_2(\alpha, i)$ implies, by Definition 5.3a, that there is an element $\underline{a}$ of $B_2(\alpha, i)$ which satisfies $\underline{a} < \underline{b}$. If $\underline{a} \notin L_1$, then $L_1 \neq L_2$, while if $\underline{a} \in L_1$, there is an element $\underline{c}$ in $B_1(\underline{a}_i, i)$ such that $\underline{c} \leq \underline{a}$. Since $\underline{a}_i = \alpha$, $\underline{c} \in B_1(\alpha, i)$. This is a violation of property 2 of Definition 5.1, since $\underline{c} \leq \underline{a} < \underline{b}$, where $\underline{b}$ and $\underline{c}$ are both elements of $B_1(\alpha, i)$. Thus $L_1 \neq L_2$, which is the desired assertion.

The third assertion of the theorem is that the process defined by Definition 5.3b, applied to a set L which has property I, yields a representation. Condition a of Definition 5.3b and condition 4 of property I show that the set S of Definition 5.3b has the properties specified in Definition 5.1.

If $(\alpha, i) \in S$, then there is at least one element $\underline{b}$ in L such that $\underline{b}_i = \alpha$. Property 4 of Definition 4.1 implies that for each element $\underline{b}$ in L for which $\underline{b}_i \geq \alpha$, there is at least one element $\underline{b}^*$ in L for which $\underline{b}^* \leq \underline{b}$ and $\underline{b}_i^* = \alpha$.

This implies that each minimal element $\underline{b}^*$ of the subset of L for which $\underline{b}_i \geq \alpha$, satisfies $\underline{b}_i^* = \alpha$, which is property 1 of Definition 5.1. From Definition 2.1, the statement that the members of $B(\alpha, i)$ are minimal elements of some set implies that no two are ordered.

The verification of property 3 of Definition 5.1 depends upon condition 4 of property I. If $\underline{a} \in B(\alpha, i)$, then $\underline{a} \in L$, and $\underline{a}$ is a minimal element of the subset $\{\underline{b}\}$ of L for which $\underline{b}_i \geq \alpha$. Thus any element $\hat{\underline{a}}$ which is covered by $\underline{a}$ satisfies $\hat{\underline{a}}_i = \underline{a}_i - 1$ and $\hat{\underline{a}}_j = \underline{a}_j$ for each j in $I(1, n)$ which is different from i . For each j such that $\hat{\underline{a}}_j = \underline{a}_j > 0$, there is an element $\underline{b}(j)$ in $B(\underline{a}_j, j)$ such that $\underline{b}(j) \leq \hat{\underline{a}} < \underline{a}$, from Definition 5.3b. This is property 3a of Definition 5.1. From the definition of minimal element, there is a member $\underline{a}^*$ of $B(\underline{a}_i - 1, i)$ for which $\underline{a}^* \leq \hat{\underline{a}} < \underline{a}$, whenever $\underline{a}_i - 1 = \alpha - 1 > 0$. This completes the proof that $\{S, B(S)\}(L)$ is a representation.

The fourth assertion is that $L(\{S, B(S)\}(L^*)) = L^*$. If L^* has property I, it is to be shown that $\underline{a} \in L^*$ if and only if $\underline{a} \in L = L(\{S, B(S)\}(L^*))$. If $\underline{a} \in L^*$, then for each i in $I(1, n)$ such that $\underline{a}_i > 0$, there is an element $\underline{a}(i)$ in $B(\underline{a}_i, i)$ such that $\underline{a} \geq \underline{a}(i)$. Lemma 5.2 shows that $\underline{a}(i) \in L$ for each such i . Since L is closed under $\vee$ by Lemma 5.3, $\vee \underline{a}(i)$ is in L , where the union is taken over all i for which $\underline{a}_i > 0$. By definition, $\underline{a} \geq \vee \underline{a}(i)$, and $\underline{a}_i = \underline{a}_i(i)$ for each i for which $\underline{a}_i > 0$, so that $\underline{a} = \vee \underline{a}(i)$ and hence is in L .

If $\underline{a} \in L$, then $\underline{a} \in [I(0, \infty)]^n$ such that for each i in $I(1, n)$ for which $\underline{a}_i > 0$, $(\underline{a}_i, i) \in S$ and there is some element $\underline{b}(i)$ in $B(\underline{a}_i, i)$ such that $\underline{a} \geq \underline{b}(i)$. Definition 5.3b implies that there is some element $\underline{d}$ in L^* such that $\underline{d}_i = \underline{b}_i(i) = \underline{a}_i$, and such that $\underline{b}(i)$ is a minimal element of the subset $\{\underline{d}\}$ of L^* for which $\underline{d}_i = \underline{a}_i$. Since $\underline{b}(i)$ is in L^* and L^* is closed under $\vee$, $\vee \underline{b}(i)$ is in L^* , where the union is taken over all i for which $\underline{a}_i > 0$. Since $(\vee \underline{b}(i))_i = \underline{a}_i$ for each such i , $\underline{a}$ is an element of L^* , and L and L^* are identical.

Thus far it has been shown that if $L(\{S_1, B_1(S_1)\}) = L(\{S_2, B_2(S_2)\})$, then

$\{S_1, B_1(S_1)\} = \{S_2, B_2(S_2)\}$ and $L(\{S, B(S)\} (L_1)) = L_1$. The next step is to show that if $\{S, B(S)\} (L_1) = \{S, B(S)\} (L_2)$ then $L_1 = L_2$. This hypothesis implies that $L(\{S, B(S)\} (L_1)) = L(\{S, B(S)\} (L_2))$, which by the preceding result implies that $L_1 = L_2$. The final assertion is that $\{S_1, B_1(S_1)\} = \{S, B(S)\} (L(\{S_1, B_1(S_1)\}))$. If $\{S, B(S)\} (L(\{S_1, B_1(S_1)\})) = \{S_2, B_2(S_2)\}$, then the statement that $L(\{S, B(S)\} (L_1)) = L_1$ implies that $L(\{S_1, B_1(S_1)\}) = L(\{S_2, B_2(S_2)\})$. This implies that $\{S_1, B_1(S_1)\} = \{S_2, B_2(S_2)\}$, and verifies the final assertion of the theorem.

The B-representation $\{S, B(S)\}$ yields a simple method of characterizing a distributive lattice which has property I. First Lemmas 5.4 and 5.5 show the equivalence of modularity and distributivity in lattices which have property I. These lemmas are derived from [7].

Lemma 5.4. If L has property I and is modular, then for any $\underline{a}$ and $\underline{b}$ in L , $\underline{a} \wedge \underline{b} \in L$.

Proof. From Definition 1.4, modularity is the property that if $\underline{a}$, $\underline{x}$ and $\underline{y}$ are distinct elements of L such that $\underline{a}$ covers $\underline{x}$ and $\underline{a}$ covers $\underline{y}$ then $\underline{x}$ and $\underline{y}$ cover $\underline{x} \cap \underline{y}$. The dual statement is also required, but is implied by property I.

If $\underline{a}$ and $\underline{b}$ are elements of L for which $\underline{a} \wedge \underline{b}$ is not in L , then for some i in $I(1, n)$, $(\underline{a} \cap \underline{b})_i < \min(\underline{a}_i, \underline{b}_i)$. Without loss of generality let $\underline{a}$ satisfy $\underline{a}_i \leq \underline{b}_i$. Then there exists a covering sequence $\underline{a} \cap \underline{b}, \underline{a}(1), \underline{a}(2), \dots, \underline{a}$, and let j be the integer such that $\underline{a}_i(j) = (\underline{a} \cap \underline{b})_i$ but $\underline{a}_i(j+1) \neq (\underline{a} \cap \underline{b})_i$. Since $\underline{a}(j)$ and $\underline{a}(j+1)$ are elements of the covering sequence, $\underline{a} \cap \underline{b} \leq \underline{a}(j) < \underline{a}(j+1) \leq \underline{a}$, and $\underline{a} \cap \underline{b} \cap \underline{b} \leq \underline{a}(j) \cap \underline{b} < \underline{a}(j+1) \cap \underline{b} \leq \underline{a} \cap \underline{b}$. From the assumption that $\underline{a}_i \leq \underline{b}_i$, the elements $\underline{a}(j)$ and $\underline{a}(j+1)$ satisfy $\underline{a}_i(j) < \underline{a}_i(j+1) \leq \underline{b}_i$, and $\underline{a}(j) \vee \underline{b} = \underline{a}(j+1) \vee \underline{b}$. These statements imply that $\underline{a}(j) = \underline{a}(j) \vee (\underline{b} \cap \underline{a}(j))$

$= \underline{a}(j) \vee (\underline{b} \wedge \underline{a}(j+1))$, and $\underline{a}(j+1) = (\underline{a}(j+1) \vee \underline{b}) \wedge \underline{a}(j+1) = (\underline{a}(j) \vee \underline{b}) \wedge \underline{a}(j+1)$, and the modular identity $\underline{a}(j) \vee (\underline{b} \wedge \underline{a}(j+1)) = (\underline{a}(j) \vee \underline{b}) \wedge \underline{a}(j+1)$ fails. Thus if L is modular and has property I, then $\underline{a} \wedge \underline{b} \in L$.

Lemma 5.5. If L has property I, L is distributive if and only if it is modular.

Proof. Since any distributive lattice is modular, from [8], it suffices to prove the reverse assertion. This is immediate from the properties of the integers and the componentwise definition of all operations.

The relation between representations and modularity in sets with property I is demonstrated in Theorem 5.3.

Theorem 5.3. A set L with property I is distributive if and only if for each (α, i) in S , $B(\alpha, i)$ has exactly one element.

Proof. From Lemma 5.5, modularity may replace distributivity in the hypothesis. To prove the sufficiency by contradiction, let L be modular and let $\underline{a}$ and $\underline{b}$ be distinct elements of $B(\alpha, i)$ for some (α, i) in S . By Lemma 5.4, $\underline{a} \wedge \underline{b} \in L$, and $(\underline{a} \wedge \underline{b})_i = \underline{a}_i = \underline{b}_i = \alpha$. This implies that $\underline{a}$ and $\underline{b}$ are not minimal elements of the subset of L for which $\underline{c}_i = \alpha$, and $\underline{a}$ and $\underline{b}$ are not in $B(\alpha, i)$. Thus $B(\alpha, i)$ has no more than one element, and is non-void by Definition 5.1. This is the sufficiency condition.

Let $B(\alpha, i)$ have exactly one element for each (α, i) in S , and let $\underline{a}$, $\underline{b}$, and $\underline{c}$ be elements of L for which $\underline{c} = \underline{a} + \underline{e}(i) = \underline{b} + \underline{e}(j)$. For each k in $I(1, n)$ distinct from i and j , the element $\underline{d}(k)$ of $B(\underline{a}_k, k)$ satisfies $\underline{d}(k) \leq \underline{a}$, $\underline{b}$, $\underline{c}$, so that $\underline{d}_i(k) \leq \underline{a}_i$ and $\underline{d}_j(k) \leq \underline{b}_j$. Thus each $\underline{d}(k)$ satisfies the relations $\underline{d}(k) \leq \underline{a} - \underline{e}(j) = \underline{b} - \underline{e}(i)$ in $[I(0, \infty)]^n$.

Let $\underline{a}(i)$, $\underline{b}(j)$, $\underline{c}(i)$ be the elements of $B(\underline{a}_i, i)$, $B(\underline{b}_j, j)$, $B(\underline{c}_i, i)$ and $B(\underline{c}_j, j)$ respectively. Since $\underline{b}_i = \underline{c}_i$, 3a of Definition 5.1 implies that $\underline{c}(i) \leq \underline{b} < \underline{c}$. Since $\underline{a}_i(i) < \underline{c}_i(i)$, $\underline{a}(i) < \underline{c}(i)$ by 3b of Definition 5.1. These two statements and the observation that $\underline{c}_j(i) \leq \underline{b}_j < \underline{a}_j$ imply that $\underline{a}(i) \leq \underline{a} - \underline{e}(j)$, where $\underline{a}(i)$ and $\underline{a} - \underline{e}(j)$ are considered as elements of $[I(0, \infty)]^n$. The corresponding statement $\underline{b}(j) \leq \underline{b} - \underline{e}(i)$ holds by symmetry. Thus $\underline{d} = \bigvee_{k \neq i, j} \underline{d}(k)$ $\underline{a}(i)$ $\underline{b}(j)$ is in L , and $\underline{d} \leq \underline{a} - \underline{e}(j) = \underline{b} - \underline{e}(i)$. From the definition of $\underline{d}(k)$, $\underline{a}(i)$ and $\underline{b}(j)$, $\underline{d} = \underline{a} - \underline{e}(j) = \underline{b} - \underline{e}(i)$, so that $\underline{d} = \underline{a} \wedge \underline{b}$, $\underline{a}$ covers $\underline{a} \wedge \underline{b}$ and $\underline{b}$ covers $\underline{a} \wedge \underline{b}$. This shows that L is modular and distributive.

Several examples illustrate the concept of B-representation. In Figure 8, page 74, the elements of S are $(1,1)$ $(\alpha,2)$ and $(\alpha,3)$ for all α in $I(1, \infty)$. The sets $B(\alpha, i)$ are defined by $B(1,1) = 100$, $B(1,2) = 010$, $B(\alpha,2) = (1\alpha 0, 0\alpha\alpha)$ when $\alpha \geq 1$, and $B(\alpha,3) = (10\alpha, 0\alpha\alpha)$, when $\alpha > 0$.

In Figure 8, the elements of $B(S)$ are underlined. The dotted line separates the two periodic classes of L . The left-hand periodic class contains two $\mathcal{L}$ -classes, the other, one. A set of functions which generate L is the following:

$$\begin{aligned} z'_1 &= 1 \\ z'_2 &= \bar{z}_1 \bar{z}_3 \vee z_1 \bar{z}_2 \\ z'_3 &= \bar{z}_1 z_2 \vee z_1 \bar{z}_3, \end{aligned}$$

where the initial state u is 000.

The second example is derived from the first by deleting elements of the form $(1\alpha 0)$ from $B(\alpha,2)$, while leaving the other elements of $B(S)$ unchanged. The set is shown in Figure 9 on page 74. Since $\{S, B(S)\}$ is a representation,

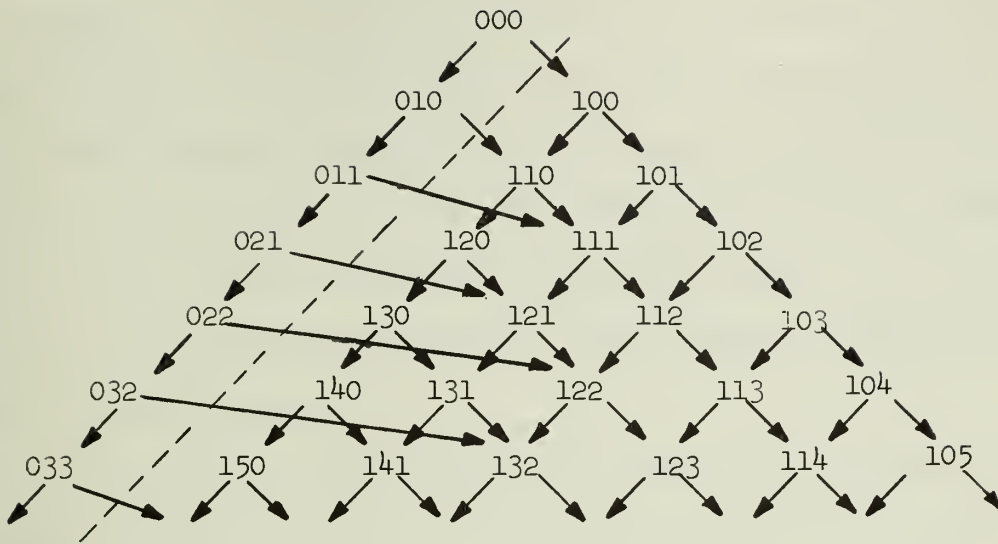


Figure 8

B-Representation of a C-State Lattice

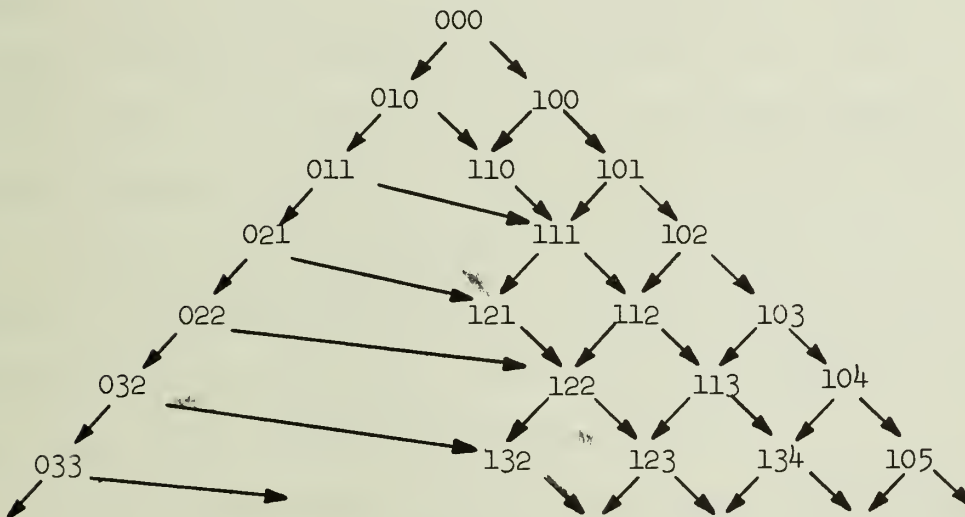


Figure 9

A B-Representation Which Does Not Have Property II

L has property I. The elements 10α and 10β are in different $\mathcal{S}$ -classes whenever $\alpha \neq \beta$, since $1\alpha + 1\beta \in L$ for $\beta \geq \alpha$, but $1\alpha + 1\beta \notin L$ when $\beta < \alpha$. Thus the number of $\mathcal{S}$ -classes of L is infinite, and L does not have property II.

These examples emphasize the importance of property II. At present the set L must be constructed to verify that property II holds. A counterpart to property II which does not necessitate the construction of L is obviously desirable. Since the elements of $B(S)$ have a finite number of $\mathcal{S}$ -classes, this condition is not sufficient.

In this section the B-representation has been developed and employed in several theorems and examples. In the next section, this representation plays an important role in the investigation of circuits with delays.

II. THE DELAY PROBLEM

In passing from physical switching circuits to mathematical models such as the semi-modular theory certain simplifying assumptions are employed. Two assumptions which are used in the present case are: (a) any change at the output of some element is propagated instantaneously to all elements of which the specified element is an input, and (b) any such physical quantity is interpreted as the same logical signal by all elements fed by the given element.

In this section the problem of removing restriction (a) is studied, making use of the B-representation. The problem to be investigated concerns the semi-modularity of circuits which result when delay elements are introduced between elements of an otherwise semi-modular circuit.

The concept of delay employed here is based on the observation that functions of the form $z'_1 = z_j$ serve only to delay the time at which a change in the signal at node j arrives at points of which node i is an input.

Definition 5.4. Let C and C^* be circuits defined by sets of functions $\{f_h\}$ and $\{f_k^*\}$ respectively, where h ranges over $I(1,n)$ and k over $I(1,n+1)$. The circuit C^* is defined to correspond to the circuit C with a delay between nodes i and $j_1, j_2, \dots, j_r$, each distinct from i , if the sets of functions $\{f_h\}$ and $\{f_k^*\}$ satisfy the following conditions: Let $j_1, \dots, j_r = M$.

1. If $k \in I(1,n)$ and $k \notin M$, then $f_k^*(z_1^*, \dots, z_n^*, z_{n+1}^*) = f_k(z_1^*, \dots, z_n^*)$,
2. $f_{n+1}^* = z_i^*$,
3. $f_k^*(z_1^*, \dots, z_{n+1}^*) = f_k(z_1^*, \dots, z_{i-1}^*, z_{n+1}^*, z_{i+1}^*, \dots, z_n^*)$ for each k in M .

The concept of semi-modularity does not enter Definition 5.4. A part of the relation between delays, semi-modularity and distributivity is demonstrated in Theorem 5.4. The converse of Theorem 5.4 is false, however.

Theorem 5.4. Let C and C^* be two circuits as given in Definition 5.4.

Let u and u^* be two initial states in T and T^* respectively, such that $u_k^* = u_k$ for each k in $I(1,n)$ and $u_{n+1}^* = u_i^*$. If C^* is s.m. $[u^*]$, then C is s.m. $[u]$, and if C^* is distributive with respect to u^* , then C is distributive with respect to u .

Proof. Let a be any state of $T[u]$ such that there is a corresponding

state a^* in $T^*[u^*]$ for which $a_h = a_h^*$ for each h in $I(1,n)$ and $a_{n+1}^* = a_i^*$. Note that the states u and u^* constitute such a pair. Suppose that $a R b$ for some state b in $T[u]$. It is to be shown (1) that there is a corresponding state b^* in $T^*[u^*]$, and (2) that $b R a'$.

In order to show (1) construct c^* by letting $c_k^* = b_k$ for each k in

$I(1,n)$ and $c_{n+1}^* = a_i^* = a_{n+1}^*$. Then $a_k'^* = f_k^*(a_1^*, a_2^*, \dots, a_{n+1}^*) = f_k(a_1^*, a_2^*, \dots, a_n^*) = f_k(a_1, a_2, \dots, a_n) = a_k'$ for each k in $I(1,n)$. Hence $a^* R c^*$. Define b^* by

$b_k^* = c_k^*$ for each k in $I(1,n)$, and $b_{n+1}^* = c_i^* = c_{n+1}^*$. Thus $b_{n+1}^* = b_i^*$ and

$b_k = b_k^*$, and b^* bears the same relationship to b that a^* does to a , and (1)

is proved. It should also be noted that $c^* R b^*$ (although if $a_{n+1}^* = a_{n+1}'$ then c^* must equal b^* , and this is a degenerate case in which $c^* R b^*$ still holds).

To show (2), that $b R a'$, note that $a^* R c^* R a^{*'}$ and $a_k^{*'} = a_k'$ for each k in $I(1,n)$. If $c_k^* = a_k^{*'}$, then the observation that $b_k = b_k^* = c_k^*$ implies that $b_k = a_k'$. If $c_k^* \neq a_k^{*'}$, then $c_k^{*'} = a_k^{*'}$ and, since $c^* R b^* R c^{*'}$ by semi-modularity, either $b_k^* = c_k^{*'} = a_k^{*'}$ and $b_k = a_k'$, or else $b_k^{*'} = c_k^{*'}$, and hence $b_k' = a_k'$.

Thus by induction from u to all states a which follow u the semi-modularity of c has been established.

To demonstrate the distributivity of C under the assumption that C^* is distributive, the characterization of distributivity displayed in Lemma 5.6 is employed.

Definition 5.5. Let a and b be two states in T . Then $a D_i b$ is defined to mean that $a_r = b_r$ for each r in $I(1,n)$ different from i , and $a_i \neq b_i = a_i'$.

Note that if $a D_i b$, then $a R b$.

Lemma 5.6. A circuit C is distributive with respect to u if and only if for every set of three states a , b , and c in $T[u]$ such that $a D_i b$ and $a D_j c$ for some pair of distinct indices i and j , when any other node k is excited in states b and c , it is excited in state a .

Proof. Let C be distributive with respect to u , and choose a set of three states as described in the statement of the lemma. Since a is in $T[u]$, there is some partial allowed sequence from u to a and the length of this sequence is a C -state denoted by $\underline{a}$. From the definition of C -state, $\underline{b} = \underline{a} + \underline{e}(i)$ and $\underline{c} = \underline{a} + \underline{e}(j)$ are formed by adding the states b and c respectively to the sequence, and $t(\underline{b}) = b$, and $t(\underline{c}) = c$. If node k is excited in both b and c , the vectors

$\underline{b} + \underline{e}(k)$ and $\underline{c} + \underline{e}(k)$ are both elements of L . Since $\underline{a} + \underline{e}(k) = [\underline{b} + \underline{e}(k)] \wedge [\underline{c} + \underline{e}(k)]$ is a C-state, node k must be excited in state a .

To prove distributivity assume the contrary. Specifically, assume that there exists a pair of C-states $\underline{x}$ and $\underline{y}$ which are both covered by a third C-state $\underline{z}$, but that $\underline{x}$ does not cover $\underline{x} \wedge \underline{y}$. Let i and j be the distinct elements of $I(1,n)$ for which $\underline{x} + \underline{e}(i) = \underline{z}$ and $\underline{y} + \underline{e}(j) = \underline{z}$. Form covering sequences $\underline{x} \wedge \underline{y}$, $\underline{x}(1), \underline{x}(2), \dots, \underline{x}(r), \underline{x}$, and $\underline{x} \wedge \underline{y}, \underline{y}(1), \underline{y}(2), \dots, \underline{y}(r), \underline{y}$. The assumption that $\underline{x} \wedge \underline{y}$ is the greatest lower bound for $\underline{x}$ and $\underline{y}$ implies that $[\underline{x} \wedge \underline{y}] + \underline{e}(j) = \underline{x}(1)$ and $[\underline{x} \wedge \underline{y}] + \underline{e}(i) = \underline{y}(1)$. By virtue of the fact that $\underline{x}$ does not cover $\underline{x} \wedge \underline{y}$, these sequences each must have at least three members. Let k be the element of $I(1,n)$ for which $\underline{x}(1) + \underline{e}(k) = \underline{x}(2)$, (where $\underline{x}(2) = \underline{x}$ if the sequence has just three members). Then node k cannot be excited in $\underline{x} \wedge \underline{y}$ for otherwise $(\underline{x} \wedge \underline{y}) + \underline{e}(k)$ would precede $\underline{x}$ and $\underline{y}$. Hence $k \neq i$ and $k \neq j$. On the other hand, since $(\underline{x} \wedge \underline{y})_k < \underline{y}_k$ there is a first state $\underline{y}(s)$ in the sequence $\underline{x} \wedge \underline{y}$ to $\underline{y}$ such that k is excited in $\underline{y}(s)$. Hence k is not excited in $\underline{y}(s - 1)$ but is excited in $\underline{y}(s)$ and in $\underline{y}(s - 1) + \underline{e}(j) = \underline{y}(s - 1) \vee \underline{x}(1)$ by semi-modularity. Also, $\underline{y}(s) \neq \underline{y}(s - 1) + \underline{e}(j)$ for otherwise $\underline{x}(1) \leq \underline{y}$. Hence the states $t(\underline{y}(s - 1))$, $t(\underline{y}(s))$, and $t(\underline{y}(s - 1) + \underline{e}(j))$ satisfy all conditions placed on a , b , and c respectively. Although k is not excited in $t(\underline{y}(s - 1))$, it is excited in the other two states. Therefore, the second half of the lemma has been proved by contradiction.

To return to the proof of Theorem 5.4, let C satisfy the sufficiency hypothesis of Lemma 5.6. From the first part of the proof, there exist elements a^* , b^* , and c^* in $T^*[u^*]$ which are equal compontnt-wise to the corresponding elements of $T[u]$ and have $a_i^* = a_{n+1}^*$, $b_i^* = b_{n+1}^*$ and $c_i^* = c_{n+1}^*$. Furthermore, for each h in $I(1,n)$, $f_h^*(a^*) = f_h(a)$, $f_h^*(b^*) = f_h(b)$ and $f_h^*(c^*) = f_h(c)$.

If node m is excited in $\underline{a}^*$ and $\underline{b}^*$, then $\underline{a}^* + \underline{e}(m)$ and $\underline{b}^* + \underline{e}(m)$ are elements of L^* . Since C^* is distributive, $[\underline{a}^* + \underline{e}(m)] \wedge [\underline{b}^* + \underline{e}(m)] = [\underline{a}^* \wedge \underline{b}^*] + \underline{e}(m) = \underline{c}^* + \underline{e}(m)$ is in L^* . This implies that node m is excited in $\underline{c}^*$ and in c^* . From the correspondence between c^* and c , node m is excited in c if it is excited in c^* and the result has been established.

In the proof of Lemma 5.6, it has been shown that if C is non-distributive, then L contains elements $\underline{y}(s-1)$, $\underline{y}(s) = \underline{y}(s-1) + \underline{e}(h)$, $\underline{y}(s-1) \vee \underline{x}(1) = \underline{y}(s-1) + \underline{e}(j)$, $\underline{y}(s) \vee \underline{x}(1) = \underline{y}(s-1) + \underline{e}(h) + \underline{e}(j)$, $(\underline{y}(s-1) \vee \underline{x}(1)) + \underline{e}(k) = \underline{y}(s-1) + \underline{e}(j) + \underline{e}(k)$, $\underline{y}(s) + \underline{e}(k) = \underline{y}(s-1) + \underline{e}(h) + \underline{e}(k)$, and $\underline{y}(s-1) + \underline{e}(j) + \underline{e}(h) + \underline{e}(k)$. Since h, j and k are all distinct, this set of elements is closed under $\vee$ and $\wedge$, and constitutes a sublattice of L . This criterion is also useful in determining semi-modularity of a set L .

Theorem 5.5. Let C be s.m. $[u]$ and let $\{f_i\}$ be the set of functions generating C . Let the pairs of distinct integers i and j of $I(1, n)$, for which there exist elements a and b of $T[u]$ such that $a R b$, $a_h = b_h$ for each h in $I(1, n)$ distinct from i , and $f_j(a) \neq f_j(b)$, be mapped uniquely onto $I(n+1, n+m)$ for suitable m . Let the image under the mapping of the pair i and j be denoted by (ij) , and let M denote the set of all images (ij) for each i and j in $I(1, n)$.

Let $\{f_{ij}^*\}$ be a set of functions defined by the following rule:

1. $z_{ij}^* = z_i^*$ for each (ij) in M ,
2. For each j in $I(1, n)$, $f_j^*(z_1^*, z_2^*, \dots, z_{n+m}^*) = f_j(\hat{z}_1^*, \hat{z}_2^*, \dots, \hat{z}_n^*)$, where $\hat{z}_i^*$ is defined by $\hat{z}_i^* = z_i^*$ if $(ij) \notin M$, and $\hat{z}_i^* = z_{ij}^*$ if $(ij) \in M$.

Let u^* be defined by $u_h^* = u_h$ for each h in $I(1, n)$, $u_{ij}^* = u_i^*$ for each (ij) in M .

If the circuit C^* defined by $\{f_h^*\}$ is s.m. $[u^*]$, then the representation $\{S, B(S)\}$ of the C -state lattice L corresponding to C has the following property:

Let (ij) be an element of M and $\underline{a}$ be a member of $B(\alpha, i)$. If $\underline{b}$ is an element of $B(\underline{a}_j, j)$, then $\underline{b}_i = \underline{a}_i - 1$. Furthermore, $\underline{a}_i - \underline{a}_j$ is a constant for all α for which $(\alpha, i) \in S$ and for all $\underline{a}$ in $B(\alpha, i)$, where the value of this constant of 0 or 1.

Proof. To facilitate the proof of this theorem, Lemmas 5.7 and 5.8 are required. These results also are applicable to the case of Theorem 5.4, in which delays are not present in all lines.

Lemma 5.7. Let C^* be a circuit which has a delay node, whose index is denoted by (ij) , between node i and node j , and let C^* be s.m. $[u^*]$, where $u_{(ij)}^* = u_i^*$. If $a^* \in T^*[u^*]$ and $a_{(ij)}^* \neq a_i^*$, then $a_i^{*'} = a_i^*$. Furthermore, for any $\underline{a}^*$ in the C -state lattice L^* of $T^*[u^*]$, either $\underline{a}_i^* = \underline{a}_{(ij)}^*$ or $\underline{a}_i^* = \underline{a}_{(ij)}^* + 1$.

Proof. Let C^* have a delay node (ij) between nodes i and j , and let C^* be s.m. $[u^*]$, where $u_{(ij)}^* = u_i^*$. Let a^* be any element of $T^*[u^*]$ for which $a_{(ij)}^* \neq a_i^*$, and let b^* be any element of $T^*[u^*]$ for which $b_{(ij)}^* = a_{(ij)}^*$ and $a^* R b^*$. The hypothesis that C^* is semi-modular implies that $a^* R b^* R a^{*'}$, and that $b_{(ij)}^{*'} = a_{(ij)}^{*'}$ or $b_{(ij)}^* = a_{(ij)}^{*'}$. This statement and the assumption that $b_{(ij)}^* = a_{(ij)}^* \neq a_{(ij)}^{*'}$ imply that $b_{(ij)}^{*'} = a_{(ij)}^{*'}$.

The hypothesis that node (ij) is a delay element whose input is node i implies that $a_{(ij)}^{*'} = a_i^*$ and $b_{(ij)}^{*'} = b_i^*$. This statement and the last equation of the preceding paragraph imply that $b_i^* = a_i^*$. Since this relation holds for each b^* in $T^*[u^*]$ for which $a^* R b^*$ and $a_{(ij)}^* = b_{(ij)}^*$, the conclusion that $a_i^{*'} = a_i^*$ follows.

The second assertion follows from the first by an induction argument.

The vector $\underline{0}$ is an element of L^* , and $\underline{0}_i^* = \underline{0}_{(ij)}^*$. If $\underline{a}^*$ is in L^* and

$\underline{a}_i^* = \underline{a}_{(ij)}^*$, then any $\underline{b}^*$ for which $a^* R b^*$ satisfies the conclusion. If $\underline{a}_i^* = \underline{a}_{(ij)}^* + 1$, then $\underline{a}_i^* \neq \underline{a}_{(ij)}^*$, $\underline{a}_i^* = \underline{a}_i^{*'}$, and $\underline{b}_i^* = \underline{a}_i^*$. Thus either $\underline{a}_{(ij)}^* + 1 = \underline{b}_{(ij)}^*$ and $\underline{b}_{(ij)}^* = \underline{b}_i^*$, or $\underline{b}_{(ij)}^* = \underline{a}_{(ij)}^*$ and $\underline{b}_{(ij)}^* + 1 = \underline{b}^*(i)$. This proves the second assertion.

Lemma 5.8. If $\underline{a}$ and $\hat{\underline{a}}$ are elements of $B(\alpha, i)$ and $B(\alpha+1, i)$ respectively, and $\underline{a} < \hat{\underline{a}}$, then $\underline{a}_j < \hat{\underline{a}}_j$ for each j for which $(ij) \in M$.

Proof. If the lemma does not hold, then $\hat{\underline{a}}_j = \underline{a}_j$, from the hypothesis that $\hat{\underline{a}} > \underline{a}$. Let the element $\underline{b}$ in L be defined to be $\underline{a} - \underline{e}(i)$. Since $\underline{a} \in B(\alpha, i)$, the unique element of L covered by $\underline{a}$ must be $\underline{a} - \underline{e}(i)$.

Let $\{\underline{b}, \underline{a}, \underline{a}(1), \dots, \underline{a}(r-1), \hat{\underline{a}}\}$ be any covering sequence from $\underline{b}$ to $\hat{\underline{a}}$ in L , and let $\{\underline{b}^*, \underline{a}^*, \underline{a}^*(1), \dots, \underline{a}^*(r-1), \hat{\underline{a}}^*\}$ be the corresponding sequence in L^* , where $\underline{a}_h^*(k) = \underline{a}_h(k)$ for each h in $I(1, n)$ and $\underline{a}_{(hm)}^*(k) = \underline{a}_h^*(k)$ for each (hm) in M .

Let the sequence $\{\underline{b}^*, \underline{a}^*, \dots, \hat{\underline{a}}^*\}$ be refined to a covering sequence in some fashion, and let this sequence be $\{\underline{b}^*, \underline{b}^*(1), \dots, \underline{b}^*(s), \hat{\underline{a}}^*\}$. Let $\underline{b}^*(h(0))$ be the first element of the sequence for which $\hat{\underline{b}}_{(ij)}^*(h(0)) = \hat{\underline{a}}_{(ij)}^* - 1$. Since C^* and L^* are defined in terms of the functions f_k^* , the elements of the sequence for which $h > h(0)$ may be replaced by $\hat{\underline{b}}^*(h) - \underline{e}(ij)$ and $\hat{\underline{b}}^*(h(0))$ removed from the sequence. This procedure affects the input to node j only, but the value of this component is unchanged in the entire sequence. The remaining components are unaltered. Thus there is an element $\underline{g}^*$ in $B^*(\alpha+1, i)$ for which $\underline{g}_{(ij)}^* < \underline{g}_i^* - 1$. This contradicts Lemma 5.7, so that whenever $\underline{a} \in B(\alpha, i)$ and $\hat{\underline{a}} \in B(\alpha+1, i)$ and $\underline{a} < \hat{\underline{a}}$, then $\underline{a}_j < \hat{\underline{a}}_j$.

The proof of Theorem 5.5 may now be completed.

The first assertion of Theorem 5.5 follows from Lemma 5.8. If $\hat{\underline{a}}$ and $\underline{a}$ are elements of $B(\alpha-1, i)$ and $B(\alpha, i)$ respectively, $\hat{\underline{a}} < \underline{a}$, and $(ij) \in M$, then $\hat{\underline{a}}_j < \underline{a}_j$ and each element $\underline{b}$ in $B(\underline{a}_j, j)$ must satisfy $\underline{b}_i = \underline{a}_i - 1$. This is the first assertion of the theorem.

The assertion that $\underline{a}_i - \underline{a}_j$ is a constant for all $\underline{a}$ in $B(\alpha, i)$ and for each j for which $(ij) \in M$ is proved by means of the preceding result. The proof is accomplished by contradiction. Let $\underline{a}(i)$ and $\hat{\underline{a}}(i)$ be elements of $B(\alpha, i)$ and $B(\alpha+1, i)$, respectively, for which $\underline{a}(i) < \hat{\underline{a}}(i)$. Let $X = \{\underline{a}(i), \dots, \underline{a}(j), \underline{a}(k(1)), \dots, \underline{a}(k(r)), \hat{\underline{a}}(j), \hat{\underline{a}}(m(1)), \dots, \hat{\underline{a}}(m(s)), \hat{\underline{a}}(i)\}$, be a covering sequence from $\underline{a}(i)$ to $\hat{\underline{a}}(i)$. Let $\underline{a}(j)$ and $\hat{\underline{a}}(j)$ differ on node j from the elements which precede them in the sequence, where $(ij) \in M$.

The assumptions that $\hat{\underline{a}}(j) < \hat{\underline{a}}(i)$ and that $\hat{\underline{a}}(i) \in B(\alpha+1, i)$ imply that there is some $m(t)$ for which $\hat{\underline{a}}_{m(t)}(j) \neq \hat{\underline{a}}_{m(t)}(i)$ and $(jm(t)) \in M$. If this assertion does not hold, then in the portion of the sequence X which follows $\underline{a}(k(r))$, $\hat{\underline{a}}(j)$ may be deleted and $\hat{\underline{a}}(m(1)), \dots, \hat{\underline{a}}(m(s)), \hat{\underline{a}}(i)$ may each be diminished by $\underline{e}(j)$. This modification preserves the covering property, and the functional definition of the sequence is unaffected. Since $\hat{\underline{a}}(i) - \underline{e}(j)$ is in the sequence and $[\hat{\underline{a}}(i) - \underline{e}(j)]_i = \hat{\underline{a}}_i(i)$, this contradicts the assumption that $\hat{\underline{a}}(i) \in B(\alpha+1, i)$. If $m(t) = i$, then $(ji) \in M$, and the assertion contradicts Lemma 5.7. If $m(t) \neq i$, then the process may be continued by examining all elements which follow $\hat{\underline{a}}(m(t))$ in the sequence to find another element $(m(t)m(y))$ of M . The process must terminate with an element $(m(x)i)$ of M . Let Y be the sequence $\{m(t), \dots, m(x), i\}$ of indices found in this process.

Lemma 5.8 shows that in a covering sequence in L , two changes, or differences between successive elements of the sequence, on the same component j must be separated by a change on each component h for which $(jh) \in M$. In the sequence X there are two such elements $\underline{a}(j)$ and $\hat{\underline{a}}(j)$. Lemma 5.8 then implies that there is an element $\underline{a}(m(t))$ in the sequence which differs from its predecessor on node $m(t)$. This new process may be carried out for all elements of the sequence Y , terminating with an element $\underline{a}^*(i)$.

Since $\underline{a}_i(i) < \underline{a}_i^*(i) < \hat{\underline{a}}_i(i)$ by construction, this contradicts the assumption that $\underline{a}_i(i) + 1 = \hat{\underline{a}}_i(i)$. Thus two changes on such a node j must be separated by a change on node i .

Thus it has been shown that if $\underline{a}(i) \in B(\alpha, i)$, $\hat{\underline{a}}(i) \in B(\alpha+1, i)$ and $\underline{a}(i) < \hat{\underline{a}}(i)$, then $\hat{\underline{a}}_j(i) = \underline{a}_j(i) + 1$. Since $\hat{\underline{a}}_i(i) = \underline{a}_i(i) + 1$, it follows that $\hat{\underline{a}}_i(i) - \hat{\underline{a}}_j(i) = \underline{a}_i(i) - \underline{a}_j(i)$. This relation may be extended to cover all possible differences between $\hat{\underline{a}}_i(i)$ and $\underline{a}_i(i)$. Although in this proof the elements $\hat{\underline{a}}(i)$ and $\underline{a}(i)$ were assumed to be ordered, this restriction may be removed by the following observation. If $\underline{a}$ and $\underline{b}$ are distinct elements of $B(\alpha, i)$, and if h is an index for which $(hi) \in M$, then if $(\beta, h) \in S$ such that $\beta > \underline{a}_h$ and $\beta > \underline{b}_h$ then $\underline{a}_h = \underline{b}_h$. If this statement is false, then there is some h for which $\underline{a}_h \neq \underline{b}_h$ and $(hi) \in M$. Since $(\max(\underline{a}_h, \underline{b}_h) + 1, h) \in S$, a contradiction of the first assertion of this theorem has occurred.

To demonstrate that the constant value for $\underline{a}_i - \underline{a}_j$ is 0 or 1, Lemma 5.8 shows that this value cannot exceed 1. The assumption in the proof that $\underline{a}_i(i) > 0$ was not employed in the construction of the sequence from $\underline{a}(j)$ to $\underline{a}(i)$. Thus the proof also shows that two changes on node j may not precede the first change on node i . This implies that $\underline{a}_i - \underline{a}_j \geq 0$, and the theorem has been proved.

An interesting corollary to Theorem 5.5 shows that non-distributivity is limited in scope. This is a restatement of an argument of the preceding proof.

Corollary 5.1. If the hypotheses of Theorem 5.5 hold, if $\underline{a}$ and $\underline{b}$ are two elements of $B(\alpha, i)$, and if $\underline{d} \in B(\underline{d}_j, j)$ where $(ji) \in M$, $\underline{d} < \underline{a}$, and no member $\underline{g}$ of $B(\underline{d}_j, j)$ satisfies $\underline{g} < \underline{b}$, then $\underline{c}_j \leq \underline{d}_j$ for each $\underline{c}$ in L .

Proof. The first part of the conclusion of Theorem 5.5 implies that $(\underline{d}_j+1, j) \notin L$, since the element $\underline{b}$ in $B(\underline{a}_i, k)$ has $\underline{b}_j = \underline{d}_j - 1 \neq \underline{d}_j+1 - 1$.

Corollary 5.2 shows that when Theorem 5.5 holds, any periodicity vectors have all non-zero components equal to 1, and that no periodicity vector may decompose into a sum of two or more such vectors in a different periodic class.

Corollary 5.2. Let C be s.m.[u] and satisfy the conditions of Theorem 5.5. For any infinite periodic class $\underline{A}$ of L and any element $q(j, \underline{A})$ of $Q(\underline{A})$, if $q_i(j, \underline{A}) > 0$ for some i in $I(1, n)$, then $q_i(j, \underline{A}) = 1$. If $q_i(j, \underline{A}(1)) = q_i(k, \underline{A}(2)) = 1$ for two periodic classes $\underline{a}(1)$ and $\underline{a}(2)$ of L , then $q(j, \underline{A}(1)) = q(k, \underline{A}(2))$.

Proof. In order to define the candidate for the periodicity vector $q(j, \underline{A})$, the set $P(i)$ is required.

Let $P(i)$ be the maximal subset of $I(1, n)$ which contains i and has the following property: For any two distinct elements i, j of $P(i)$, there exist sequences $\{i, h(1), \dots, h(r), j\}$ and $\{j, k(1), \dots, k(s), i\}$ of elements of $P(i)$ such that each consecutive pair $h(m), h(m+1)$ or $k(m), k(m+1)$ yields an element $(h(m)h(m+1))$ or $(k(m)k(m+1))$ of M . The set $\{P(j)\}$, as j ranges over $I(1, n)$, yields an equivalence relation on $I(1, n)$, since it is reflexive, symmetric, and transitive. Let $\underline{p}(i)$ be defined to be $\sum_{j \in P(i)} \underline{e}(j)$.

The proof is composed of three parts, (a) showing that all elements of $P(i)$ are spanned if any are spanned and that the corresponding components are equal, (b) demonstrating that if $\underline{a}$ is in a periodic class $\underline{A}$ and has a periodicity vector $q(h, \underline{A})$ which spans node j , then $\underline{a} + \underline{p}(i) \in \underline{A}$, and (c) proving that $\underline{p}(i)$ is the periodicity vector of $\underline{A}$ which spans i .

If node i is spanned in $\underline{A}$, then $(\alpha, i) \in S$ for arbitrarily large α , and for each j for which $(ij) \in M$, the second assertion of Theorem 5.5 implies that node j is spanned also. This statement also implies that j is in $P(i)$. From the invariance of $\underline{a}_i - \underline{a}_j$ for $\underline{a}$ in $B(\alpha, i)$, the corresponding components of the periodicity vector must be equal. Since node j is spanned in $\underline{A}$, the same process

may be repeated for each h in $P(i)$. Since the definition of $P(i)$ is independent of the periodic class $\underline{A}$, all nodes of the set must be spanned by any vector which spans some node of $P(i)$. This demonstrates assertion (a).

If $\underline{a} \in \underline{A}$ and $q_j(h, \underline{A}) > 0$ for some j in $P(i)$, then it must be shown that $\underline{a} + \underline{p}(i) \in \underline{A}$. Consequently, let j be any element of $P(i)$ for which $q_j(h, \underline{A}) > 0$. Since $q_j(h, \underline{A}) > 0$, then $\underline{a} + mq(h, \underline{A}) \in \underline{A}$ for arbitrarily large m , and in particular, there is at least one element $\underline{a}^*(j)$ of $B(\underline{a}_j + 1, j)$ for which $\underline{a}^*(j) \leq \underline{a} + q(h, \underline{A})$, since $[\underline{a} + q(h, \underline{A})]_j \leq \underline{a}_j + 1$. The proof of Theorem 5.5 may be adapted to prove the assertion that there is some $\underline{b}(j)$ in L for which $\underline{b}_j(j) = \underline{a}_j + 1$, $\underline{b}(j) \leq \underline{a} + \underline{p}(i)$ and $\underline{a}_k = \underline{b}_k$ for each k not in $P(i)$.

Let X be a covering sequence $\{\underline{a}, \underline{a}(h(1)), \dots, \underline{a}(h(r)), \dots, \underline{a} \vee \underline{a}^*(j) (= \underline{b}(j))\}$. Since $\underline{a}^*(j) \leq \underline{a} + q(h, \underline{A})$, it follows that $\underline{a} \vee \underline{a}^*(j) \leq \underline{a} + q(h, \underline{A})$. If there is some element $\underline{a}(h(t))$ of X for which there is no sequence $Y = \{h(t), h(s), h(k), \dots, h(m), j\}$ such that for each successive pair of elements in Y , such as $h(s), h(k)$, the element $(h(s)h(k))$ is in M , and for which $\underline{a}(h(t)) < \underline{a}(h(s)) < \dots < \underline{a} \vee \underline{a}^*(j) = \underline{b}(j)$, then X may be replaced by a new sequence in which $\underline{a}(h(t))$ is deleted and all elements of the original sequence which follow $\underline{a}(h(t))$ are decreased by $\underline{e}(h(t))$. This process may be continued until each element of the sequence X is connected to the final element, $\underline{b}(j)$, by a sequence Y . At this point, the present sequence X has the same properties as the sequence X in the proof of Theorem 5.5. Thus the same conclusion, that there is no more than one change on any node in such a sequence, holds. Furthermore, all nodes which change in such a sequence must be elements of $P(i)$, so that $\underline{a} < \underline{b}(j) \leq \underline{a} + \underline{p}(i)$. Since such a result holds for each j in $P(i)$, $\bigvee_{j \in P(i)} \underline{b}(j) \in L$. Let $\underline{b}$ equal $\bigvee_{j \in P(i)} \underline{b}(j)$. Since $\underline{b}(j) \leq \underline{a} + \underline{p}(i)$, then $\underline{b} \leq \underline{a} + \underline{p}(i)$ and since $\underline{b}_j(j) = \underline{a}_j + 1$ for each such j , then $\underline{b} = \underline{a} + \underline{p}(i)$ and $\underline{a} + \underline{p}(i) \in L$. The element $\underline{b}$ is in $\underline{A}$, by Definition 3.1 of the periodic class. Thus (b) has been shown.

The final step of the proof is to show that $L(\underline{a}) = L(\underline{a} + \underline{p}(i))$. If $\underline{b} \in L(\underline{a})$, then $\underline{a} + \underline{b} \in L$, and by Theorem 2.6, all nodes spanned in $\underline{a}$ are spanned in $\underline{a} + \underline{b}$. This implies that $\underline{a} + \underline{b} + \underline{p}(i) \in L$ and $\underline{b} \in L(\underline{a} + \underline{p}(i))$. Thus $L(\underline{a}) \subseteq L(\underline{a} + \underline{p}(i)) \subseteq L(\underline{a} + 2\underline{p}(i)) \dots$. Since C is s.m.[u], property II holds, and the number of $\mathcal{A}$ -classes of L is finite. Thus in the sequence $L(\underline{a}) \subseteq L(\underline{a} + \underline{p}(i)) \dots$ there must be eventual equality between successive elements, and $\underline{p}(i)$ is a sum of periodicity vectors by Theorem 2.8. It has been shown that all nodes are spanned by the same vector, and no vector smaller than $\underline{p}(i)$ exists which spans all elements of $P(i)$. Thus $\underline{p}(i)$ is the unique periodicity vector which spans all nodes of $P(i)$.

Theorem 5.5 provides necessary conditions for placing delays in all lines. This is the most restrictive class of circuits with delays, since no information is available concerning times at which different elements become excited by the "same" signal change.

Circuits in which delays "inside" the element are neglected, and those which admit different discrimination levels are more difficult to treat, due to the many possible arrangements of delays and/or level differences. In the case of distributive circuits, there is a costly method of solving these problems.

In Chapter V, the B-representation is developed and shown to be equivalent to property I. In addition to aiding the decision and synthesis procedures, the B-representation is effective in considering the problem of delays in all lines of a circuit.

CHAPTER VI

CONCLUSIONS

In this chapter, the results of the investigation are summarized, and several areas for future investigation are discussed.

I. RESULTS OF THE INVESTIGATION

The principal result of this paper is contained in Theorem 4.5, in which the class of C-state lattices of semi-modular circuits was shown to be equivalent to the class of all sets L which have properties I and II. The mathematical tools used in this proof included the concepts of Δ -relation, periodic class, periodicity vectors and their structures, and the coefficient λ .

In preparation for proving Theorem 4.5, the "function theorem" 4.2 was developed as a general method for determining whether a given mapping of a set L onto a tentative set of states yields a semi-modular circuit.

In Chapter V, the B-representation of a lattice L which has property I was introduced. This set of join-irreducible elements of L completely specifies the set L . The B-representation in turn was employed to characterize the class of lattices which may be realized as circuits which satisfy the semi-modularity condition when arbitrary delay elements are inserted in all lines.

II. FURTHER AREAS FOR INVESTIGATION

Further research topics in semi-modular circuit theory fall into three categories. First, further study is indicated on problems of practical circuits, such as conditional sequences and a further investigation of delay and discrimination level problems. The second area involves possible extensions of the class of admissible representations, while the third class of problems includes

finding canonical representations of lattices and minimal circuits in the general case.

Problems in the design of useful circuits. The design of switching circuits, such as electronic digital computers, almost always includes conditional paths, in which some signal value indicates which of two sequences of operations are to take place. The present method for handling such problems is to formulate the requirements separately for each possible condition, derive the corresponding circuits and combine them as a final step. An extension of the theory in which all such possible sequences may be examined simultaneously would be valuable.

An investigation into circuits in which delays within elements may be ignored may reveal that Theorem 5.5 is unduly restrictive. The interchange of information between two electronic computers in different cities is one problem in which the assumption might well be relaxed. The second class of problems occurs when some physical quantity may be interpreted as different logical signals by two elements. These two problems may both be formulated in logical terms, representing the ordering between changes of logical signals on all lines connected directly to some output. A solution to the restricted form of the problem has been obtained, but the many possible variations for each circuit make the general problem very difficult.

Increasing the usefulness of representations. The B-representation was introduced in Chapter V to eliminate redundancy inherent in the specification of a synthesis problem by means of a lattice. Condition 3 of Definition 5.1 shows that this representation is redundant also. All that is logically required is to specify the covering relations involved between elements of $B(\alpha, i)$ and $B(\beta, j)$. The difficulty is in formulating requirements that sets of such covering relations yield a representation, and hence a lattice.

Although a representation $\{S, B(S)\}$ may be given as the specification for a circuit, property II must be verified from the corresponding L . The problem of replacing property II by a corresponding property of the elements of $B(S)$ seems accessible. Examples show that the requirements on such a property include an assumption concerning periodicity vectors of $B(S)$ as well as the restriction that the number of $\mathcal{L}$ -classes of $B(S)$ be finite. A conjecture is that these two conditions and an ordering between spanned components of two or more elements of any $B(\alpha, i)$ suffice.

Derivation of functions which define the circuit. The "function theorem" 4.2 may be employed to determine whether a given set $\{g_i\}$ maps L onto a set of states of a semi-modular circuit. When additional requirements are imposed upon the functions g_i , they may not satisfy the criterion. By a method similar to that of the proof of Theorem 4.4, the original set L may be embedded in a larger space such that any assignment of values of the g 's is acceptable. For example, L may be required to be binary, and the corresponding insertions of join-irreducible elements to ensure this property could be displayed. An additional advantage may be that few types of functions are necessary for this construction.

The final problem is that of finding that semi-modular circuit which satisfies a given problem specification and is best with respect to some weighing function. This question is pertinent in other theories of switching, as well as in the semi-modular theory.

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